Asymptotics for Toeplitz determinants: perturbation of symbols with a gap

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Toeplitz Matrices

We consider Toeplitz determinants of the form

$$D_n(f) = \det(f_{j-k})_{j,k=0,\dots,n-1}, \quad f_j = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ij\theta}d\theta,$$

where the symbol f is of the form

$$f(e^{i\theta}) = e^{W(e^{i\theta})} \times \begin{cases} 1, & \text{for } -\theta_0 \leq \theta \leq \theta_0, \\ s, & \text{for } \theta_0 < \theta < 2\pi - \theta_0, \end{cases}$$

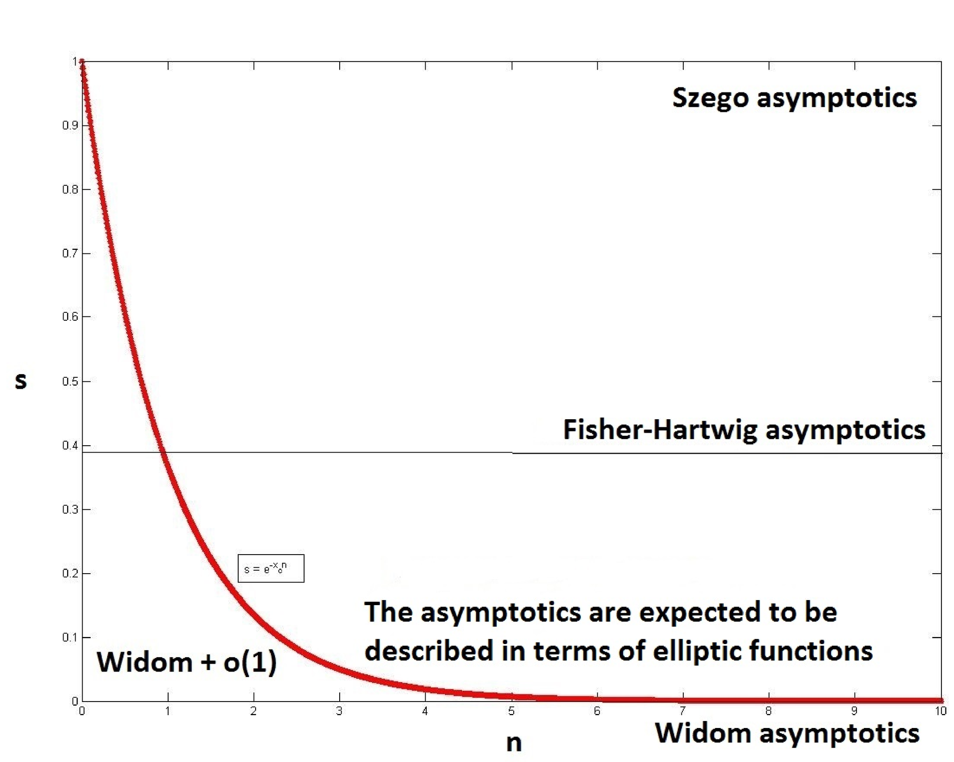
where $0 < \theta_0 < \pi$, $0 \leq s \leq 1$, and W is analytic on S_1.

Three known results

- If $s = 1$, the weight $f(e^{i\theta}) = e^{W(e^{i\theta})}$ is **analytic**. To get asymptotics, we can use the Szegő strong limit theorem:
$$\ln D_n(s=1, \theta_0, W) = nW_0 + \sum_{k=1}^{+\infty} kW_kW_{-k} + o(1),$$
as $n \to +\infty$, where $W_k = \frac{1}{2\pi}\int_0^{2\pi} W(e^{i\theta})e^{-ik\theta}d\theta$.
- If $s = 0$, the symbol is **supported on an arc**:
$$f(e^{i\theta}) = e^{W(e^{i\theta})} \times \begin{cases} 1, & \text{for } -\theta_0 \leq \theta \leq \theta_0, \\ 0, & \text{for } \theta_0 < \theta < 2\pi - \theta_0. \end{cases}$$
Asymptotics for D_n are given by (Widom, 1971)
$$\ln D_n(s=0, \theta_0, W) = n^2 \ln \sin\frac{\theta_0}{2} + nW_0 - \frac{1}{4}\ln n + \sum_{k=1}^{+\infty} kW_kW_{-k} - \frac{1}{4}\ln\cos\frac{\theta_0}{2} + \frac{1}{12}\ln 2 + 3\zeta'(-1) + o(1),$$
as $n \to +\infty$. These asymptotics are proved only if f is positive on its support and symmetric in θ.
- If s is fixed, namely independent of n, f has two jump discontinuities which are special cases of **Fisher-Hartwig singularities**. Asymptotics for D_n are given by (Deift-Its-Krasovky 2010, Widom, Basor,...)
$$\ln D_n(s, \theta_0, W) = nW_0 + \frac{(\ln s)^2}{2\pi^2}\ln n + \frac{\ln(2\sin\theta_0)}{2\pi^2} + \frac{\ln s}{\pi}\sum_{k=1}^{+\infty}(W_k + W_{-k})\sin(k\theta_0) + \sum_{k=1}^{+\infty} kW_kW_{-k} + 2\ln\left(G\left(1+\frac{\ln s}{2\pi i}\right)G\left(1-\frac{\ln s}{2\pi i}\right)\right) + o(1),$$
as $n \to +\infty$, where G is Barnes' G-function.

Critical transition

The asymptotic expansion is different for $s \to 0$ in Fisher-Hartwig asymptotics than for $s = 0$. This shows that there is critical transition when $s \to 0$, depending on the speed of convergence of s to 0.



Theorem [Charlier, Claeys 2014]

Let $\theta_0 \in (0, \pi)$ and define

$$x_c = -2\ln\tan\frac{\theta_0}{4}.$$

As $n \to \infty$ and simultaneously $s \to 0$ in such a way that $0 \leq s \leq e^{-x_c n}$, we have

$$\ln D_n(s, \theta_0, W) = \ln D_n(0, \theta_0, W) + o(1).$$

If $\epsilon < \theta_0 < \pi - \epsilon$,

$$o(1) = \mathcal{O}(n^{-1/2}se^{x_c n}).$$

We also generalise the result to the case where $\theta_0 \to \pi$, $\pi - \theta_0 > \frac{M}{n}$ with M suffiently large. Here

$$o(1) = \mathcal{O}((\pi - \theta_0)^{1/2}n^{-1/2}se^{x_c n}).$$

Applications

- **The Circular Unitary Ensemble.** Define the random variable X as the number of eigenvalues of a CUE matrix on the arc $\gamma = \{e^{i\theta} : -\theta_0 \leq \theta \leq \theta_0\}$. The moment generating function of X can be written in terms of a Toeplitz determinant:
$$F(\lambda) = \mathbb{E}_n(e^{\lambda X}) = e^{n\lambda}D_n(s = e^{-\lambda}, \theta_0, W = 0).$$
- **The sine kernel determinants (Bothner, Deift, Its, Krasovsky 2014).** In the double scaling limit $n \to +\infty$ and $\pi \to \theta_0$ at a rate $\mathcal{O}(n^{-1})$, the Toeplitz determinant D_n converges to a Fredholm determinant.

The proof

Differential identity

$$\partial_s \ln D_n(s, \theta_0, W) = \frac{1}{2\pi i}\int_{S_1} z^{-n}\left[Y^{-1}(z)Y'(z)\right]_{21}\dot{f}(z,s)dz,$$

where Y is the unique solution of a RHP for orthogonal polynomials on the unit circle.

- We want to obtain asymptotics for Y using the steepest descent analysis.
- This requires transformations on Y to get an "easy" RHP.
- The first transformation uses the equilibrium measure $u(e^{i\theta})d\theta$.

Equilibrium measure

We introduce the new parameter x: $s = e^{-xn}$

- For $x \geq x_c$,
$$u_x(e^{i\theta}) = \frac{1}{2\pi}\sqrt{\frac{\cos\theta + 1}{\cos\theta - \cos\theta_0}}$$
and its support is $\gamma = \{e^{i\theta} : -\theta_0 \leq \theta \leq \theta_0\}$. If $x \neq x_c$, it is "regular" on $S_1 \setminus \gamma$. If $x = x_c$, it is "regular" on $S_1 \setminus (\gamma \cup \{-1\})$, and it is "non-regular" on -1.

Steepest descent analysis:

- Normalisation at infinity.
- Opening of the lenses.
- Global Parametrix.
- Local parametrices near $e^{i\theta_0}$, $e^{-i\theta_0}$, with 4 branches using Bessel functions.
- Local parametrix near -1.
- Final transformation.

Integration of the differential identity:

We started integration from 0:

$$\ln D_n(s_0, \theta_0, W) = \ln D_n(0, \theta_0, W) + \int_0^{s_0}\partial_s \ln D_n(s, \theta_0, W)ds$$

The integral can be bounded using Y asymptotics.

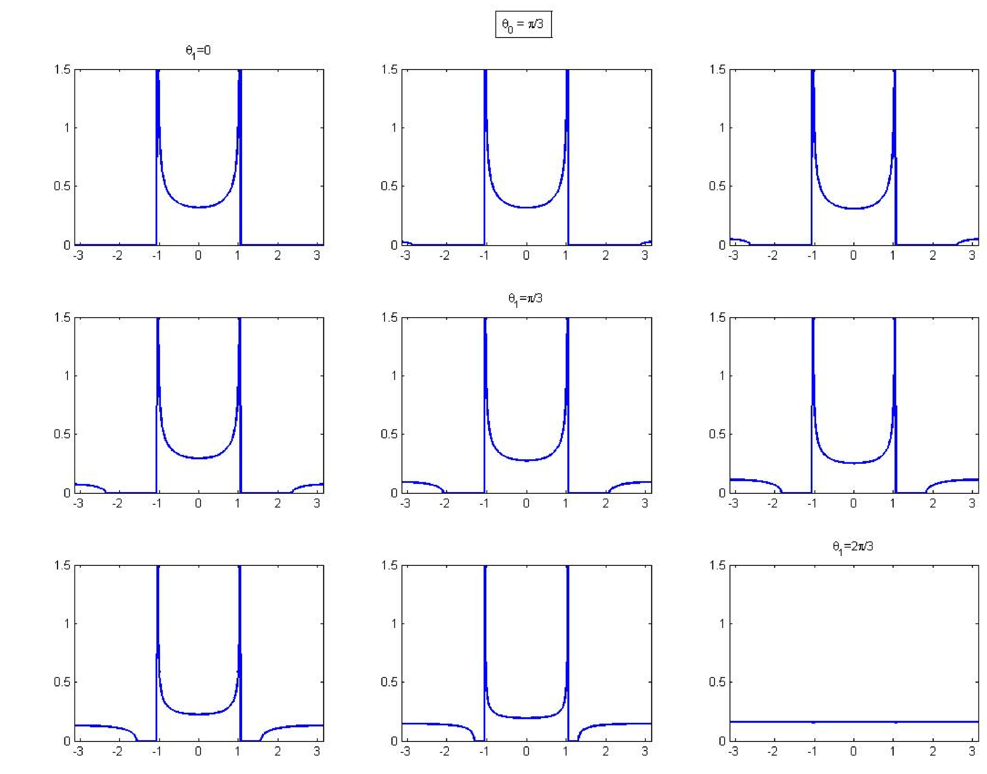
Further developments

The case $s > e^{-x_c n}$

RH analysis can be done:

Equilibrium measure

- For $0 < x < x_c$, the support of the equilibrium measure consists of **two** disjoint arcs: we have
$$J_x = \gamma \cup \{e^{i\theta} : \theta \in [\pi - \theta_1, \pi + \theta_1]\},$$
where $\theta_1 \in [0, \pi - \theta_0]$ is the unique solution of
$$2\int_{[-\theta_0,\theta_0]\cup[\pi-\theta_1,\pi+\theta_1]} \log\left|\frac{1+e^{i\theta}}{1-e^{i\theta}}\right| u_x(e^{i\theta})d\theta = x,$$
and the density is given by
$$u_x(e^{i\theta}) = \frac{1}{2\pi}\sqrt{\frac{\cos\theta + \cos\theta_1}{\cos\theta - \cos\theta_0}}.$$
The equilibrium measure is "regular" on $S_1 \setminus J_x$



- The global parametrix involves elliptic θ functions.
- Local parametrices near $e^{i(\pi-\theta_1)}$, $e^{i(\pi+\theta_1)}$ using Airy functions.
- We still need to integrate the differential identity.

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arxiv:math/1409.0435