Lyapunov exponents for products of Ginibre matrices with $\beta = 1, 2, 4$

Jesper R. Ipsen

Department of Physics, Bielefeld University

Universität Bielefeld

Expressions for a product of M Ginibre matrices

We are interested in the spectral properties of the following product matrix,

$$Y(M) \equiv X_M X_{M-1} \cdots X_1,$$

where X_m ($m = 1, \ldots, M$) are independent $(N + \nu_m) \times (N + \nu_{m-1})$ rectangular Ginibre matrices, i.e. matrices where the entries are independent centred Gaussian random variables. Here N is a positive integer denoting the smallest matrix dimension, while ν_m are non-negative integers incorporating the rectangular structure. We choose the ordering $0 = \nu_0 \leq \nu_1 \leq \cdots \leq \nu_M$ without loss of generality, see Ipsen & Kieburg (2014). For simplicity we will also assume that there is an upper bound for the rectangularity $\nu_M \to \nu_\infty < \infty$. We use the standard classification and consider three distinct classes: real ($\beta = 1$), complex ($\beta = 2$), and quaternionic ($\beta = 4$) matrices.

Given a product of M real Ginibre matrices, we know that all eigenvalues become real as M tends to infinity, see Lakshminarayan (2013) and Forrester (2014). If we explicitly assume that all eigenvalues are real (which is an excellent approximation even at moderate M), then we may write the joint density function for the eigenvalues as

$$P_N^{\beta=1}(x_1, \ldots, x_N; M) = \frac{1}{Z_{N,\nu}^{\beta=1}(M)} \prod_{1 \leq k < \ell \leq N} |x_k - x_\ell| \prod_{n=1}^{N} w_\nu^{\beta=1}(x_n),$$

$$Z_{N,\nu}^{\beta=1}(M) = N!\, 2^{MN(N+1)/4} \prod_{n=1}^{N} \prod_{m=1}^{M} \Gamma\Big[\frac{\nu_m}{2} + \frac{n}{2}\Big],$$

where $w_\nu^\beta(x_n)$ is the weight function (given below) and the constant $Z_{N,\nu}^{\beta=1}(M)$ is chosen such that integration over the eigenvalues yields the probability that all eigenvalues are real, see Forrester (2014) and Ipsen & Kieburg (2014).

The joint density functions for the eigenvalues of complex ($\beta = 2$) and quaternionic ($\beta = 4$) matrices may be written down without any assumptions,

$$P_N^{\beta=2}(z_1, \ldots, z_N; M) = \frac{1}{Z_{N,\nu}^{\beta=2}(M)} \prod_{1 \leq k < \ell \leq N} |z_k - z_\ell|^2 \prod_{n=1}^{N} w_\nu^{\beta=2}(z_n)$$

$$P_N^{\beta=4}(z_1, \ldots, z_N; M) = \frac{1}{Z_{N,\nu}^{\beta=4}(M)} \prod_{1 \leq k < \ell \leq N} |z_k - z_\ell|^2 |z_k - z_\ell^*|^2 \prod_{n=1}^{N} |z_n - z_n^*|^2 w_\nu^{\beta=4}(z_n),$$

see Akemann & Burda (2012), Ipsen (2013) and Ipsen & Kieburg (2014). The normalization constants $Z_{N,\nu}^{\beta=2}(M)$ and $Z_{N,\nu}^{\beta=4}(M)$ are given by

$$Z_{N,\nu}^{\beta=2}(M) = N!\, \pi^N \prod_{n=1}^{N} \prod_{m=1}^{M} \Gamma[\nu_m + n] \quad \text{and} \quad Z_{N,\nu}^{\beta=4}(M) = \frac{N!\, \pi^N}{2^{MN(N+1)}} \prod_{n=1}^{N} \prod_{m=1}^{M} \Gamma[2\nu_m + 2n].$$

Note that in these cases ($\beta = 2, 4$) integration over the eigenvalues yields unity for any M.

The weight functions for all three joint densities may be written collectively as a Meijer G-function,

$$w_\nu^\beta(z) = G_{0,M}^{M,0}\left({- \atop \frac{\beta\nu_1}{2}, \ldots, \frac{\beta\nu_M}{2}} \,\middle|\, \Big(\frac{\beta}{2}\Big)^M |z|^2 \right).$$

Lyapunov exponents for products of Ginibre matrices

If $z_n(M)$ ($n = 1, \ldots, N$) denote the eigenvalues of the product matrix, then we define the finite-M Lyapunov exponents by

$$\lambda_n(M) \equiv \frac{\log |z_n(M)|}{M},$$

where $|z_n(M)|$ are the moduli of the eigenvalues. We introduce the joint density function of the finite-M Lyapunov exponents

$$\rho_N^{\beta=1}(\lambda_1, \pm, \ldots, \lambda_N, \pm; M) \equiv \prod_{n=1}^{N} M e^{M\lambda_n}\, P_N^{\beta=1}(\pm e^{M\lambda_1}, \ldots, \pm e^{M\lambda_N}; M),$$

$$\rho_N^{\beta=2,4}(\lambda_1, \theta_1, \ldots, \lambda_N, \theta_N; M) \equiv \prod_{n=1}^{N} M e^{2M\lambda_n}\, P_N^{\beta=2,4}(e^{M\lambda_1 + i\theta_1}, \ldots, e^{M\lambda_N + i\theta_N}; M).$$

The prefactor on the right hand side is due to the change of variables. For a large number of matrices ($M \gg 1$) the Lyapunov exponents become exponentially separated. As a consequence, the mutual interaction between eigenvalues becomes neglectable and the joint densities turn into permanental point processes,

$$\rho_N^{\beta=1}(\lambda_1, \pm, \ldots, \lambda_N, \pm; M \gg 1) \approx \frac{1}{N!} \operatorname*{per}_{1 \leq k,\ell \leq N} \Big[\frac{1}{2} f_k^{\beta=1}(\lambda_\ell; M \gg 1)\Big],$$

$$\rho_N^{\beta=2}(\lambda_1, \theta_1, \ldots, \lambda_N, \theta_N; M \gg 1) \approx \frac{1}{N!} \operatorname*{per}_{1 \leq k,\ell \leq N} \Big[\frac{1}{2\pi} f_k^{\beta=2}(\lambda_\ell; M \gg 1)\Big],$$

$$\rho_N^{\beta=4}(\lambda_1, \theta_1, \ldots, \lambda_N, \theta_N; M \gg 1) \approx \frac{1}{N!} \operatorname*{per}_{1 \leq k,\ell \leq N} \Big[\frac{2\sin^2\theta_\ell}{\pi} f_k^{\beta=4}(\lambda_\ell; M \gg 1)\Big]$$

for real, complex and quaternionic matrices, respectively. Here "per" denotes the permanent and

$$f_n^\beta(\lambda; M \gg 1) = \frac{1}{\sqrt{2\pi}\sigma_n^\beta} \exp\Big[-\frac{(\lambda - \mu_n^\beta)^2}{2(\sigma_n^\beta)^2}\Big],$$

where the mean and variance is given by

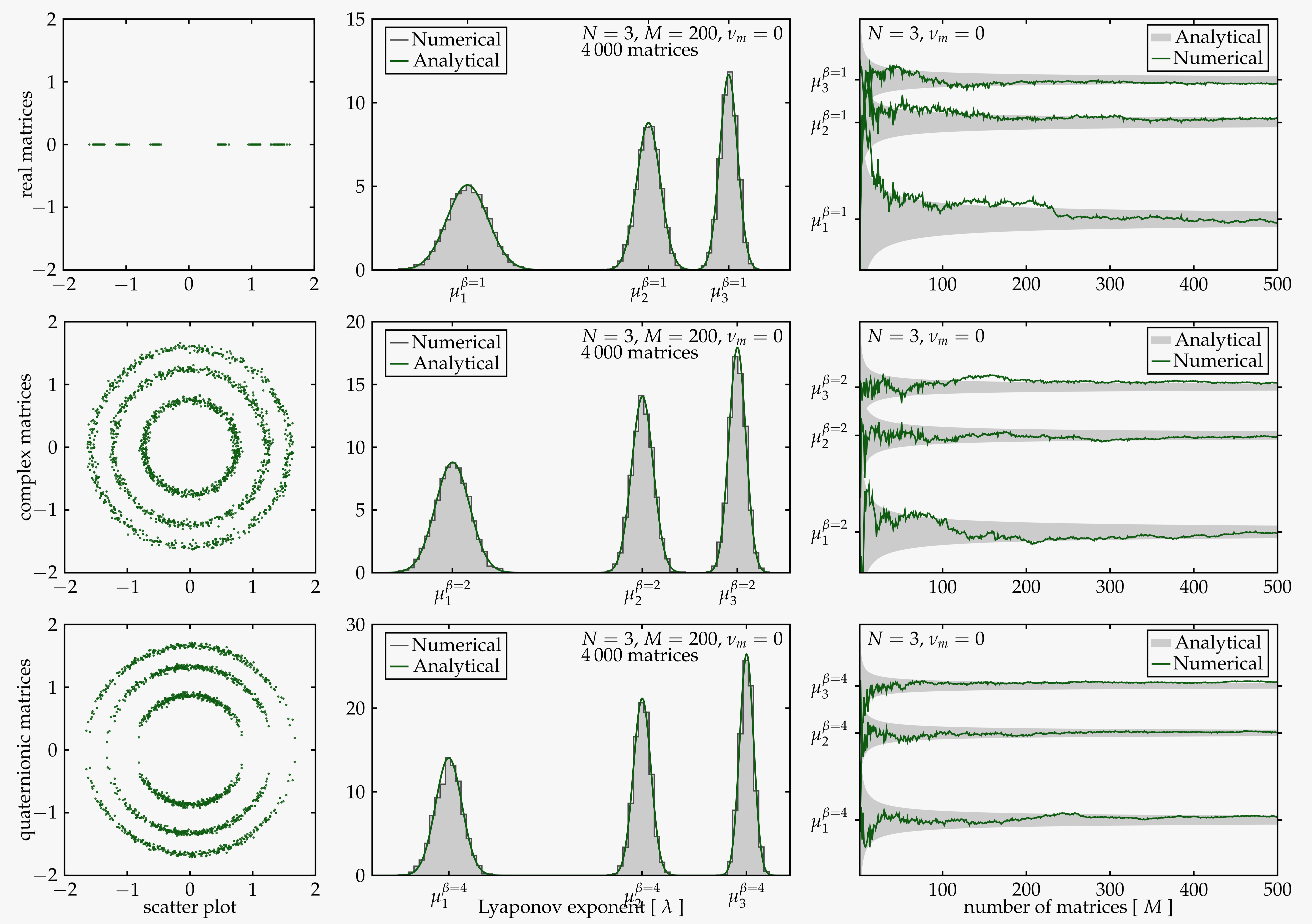
$$\mu_n^\beta = \frac{1}{2}\log\frac{2}{\beta} + \frac{1}{2}\Big\langle \psi^{(0)}\Big(\frac{\beta(\nu + n)}{2}\Big)\Big\rangle \quad \text{and} \quad (\sigma_n^\beta)^2 = \frac{1}{4M}\Big\langle \psi^{(1)}\Big(\frac{\beta(\nu + n)}{2}\Big)\Big\rangle,$$

respectively. Here $\psi^{(0)}(x) \equiv \Gamma'[x]/\Gamma[x]$ is the digamma function, $\psi^{(1)}(x)$ is the first derivative of digamma function. The average is defined as

$$\Big\langle \psi^{(i)}\Big(\frac{\beta(\nu + n)}{2}\Big)\Big\rangle \equiv \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \psi^{(i)}\Big(\frac{\beta(\nu_m + n)}{2}\Big), \quad i = 0, 1.$$

Recall that $n = 1, \ldots, N$ and $\nu_M \to \nu_\infty < \infty$ which ensures that the limit is well-defined without further rescaling.

For a product of square, complex Ginibre matrices this result was previously obtained by Akemann, Burda & Kieburg (2014). If we consider square matrices ($\nu_m = 0$ for all m) then the average becomes trivial, $\langle \psi^{(i)}(\beta n/2) \rangle = \psi^{(i)}(\beta n/2)$, in agreement with the aforementioned result.



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More on products at this workshop

Poster by T. Neuschel, *Jacobi polynomial moments and products of random matrices*

Poster by D. Stivigny, *Products with truncated unitary matrices*

Talk by K. Życzkowski tomorrow, *Random density matrices and convolutions of Marchenko-Pastur distribution*

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