

Fluctuations of Linear Eigenvalue Statistics of Random Band Matrices

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joint work with

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Definitions

Define a real symmetric random band matrix $M = (m_{jk})_{n \times n}$ of bandwidth b_n as

$$m_{jk} = m_{kj} = \begin{cases} b_n^{-1/2} w_{jk} & \text{if } d_n(j, k) \leq b_n \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where $d_n(j, k) := \min\{|j - k|, n - |j - k|\}$ and $\{w_{jk}\}_{j \leq k}$ is a sequence of independent real random variables with

$$\mathbb{E}[w_{jk}] = 0, \quad \mathbb{E}[w_{jk}^2] = \begin{cases} 1 & \text{if } j \neq k \\ \sigma^2 & \text{if } j = k. \end{cases} \quad (2)$$

Definitions

Let $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ be the eigenvalues of the random band matrix M . Define the linear eigenvalue statistic of the eigenvalues of M as

$$\mathcal{N}_n(\phi) = \sum_{i=1}^n \phi(\lambda_i), \quad (3)$$

and the normalized eigenvalue statistic of the matrix M as

$$\mathcal{M}_n(\phi) = \sqrt{\frac{b_n}{n}} \mathcal{N}_n(\phi), \quad (4)$$

where ϕ is a test function.

Main Theorem

Theorem

Let M be a real symmetric random band matrix as defined in (1), and b_n be a sequence of integers satisfying $\sqrt{n} \ll b_n \ll n$. Assume the following:

- (i) The probability distribution of w_{jk} satisfies the Poincaré inequality with some uniform constant m which does not depend on n, j, k .
- (ii) $\mathbb{E}[w_{jk}^4] = \mu_4$ for all $d_n(j, k) \leq b_n$.
- (iii) $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a test function in the Sobolev space H^s i.e., $\|\phi\|_s < \infty$, where

$$\|\phi\|_s^2 = \int_{\mathbb{R}} (1 + 2|t|)^{2s} |\hat{\phi}(t)|^2 dt,$$
$$\hat{\phi}(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-it\lambda} \phi(\lambda) d\lambda,$$

and $s > 5/2$.

Main Theorem

Then the centred normalized eigenvalue statistic $\mathcal{M}^\circ(\phi) = \mathcal{M}_n(\phi) - \mathbb{E}[\mathcal{M}_n(\phi)]$ converges in distribution to the Gaussian random variable with mean zero and variance given by

$$V(\phi) = \frac{\kappa_4}{16\pi^2} \left(\int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{4 - \mu^2}{\sqrt{8 - \mu^2}} \phi(\mu) d\mu \right)^2 + \frac{\sigma^2}{16\pi^2} \left(\int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\mu\phi(\mu)}{\sqrt{8 - \mu^2}} d\mu \right)^2$$
$$+ \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{(8 - x^2)(8 - y^2)} F(x, y) \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\mu_1\phi(\mu_1)}{(x - \mu_1)\sqrt{8 - \mu_1^2}} \frac{\mu_2\phi(\mu_2)}{(x - \mu_2)^2\sqrt{8 - \mu_2^2}} d\mu_1 d\mu_2 dx dy,$$

where for $x \neq y$

$$F(x, y) = \sqrt{2} \int_{-\infty}^{\infty} \frac{(s^3 \sin s - s \sin^3 s)}{2(s^2 - \sin^2 s)^2 - (s^3 \sin s + s \sin^3 s)xy + s^2 \sin^2 s(x^2 + y^2)} ds,$$

and κ_4 is the fourth cumulant of the off-diagonal entries, i.e., $\kappa_4 = \mathbb{E}[W_{12}^4] - 3$.

Proof | Step 1 | Approximate ϕ by smooth functions

Define $e_n(x) = e^{ix\mathcal{M}_n^\circ(\phi)}$, and $Z_n(x) = \mathbb{E}[e_n(x)]$. We want to show that

$$\lim_{n \rightarrow \infty} Z_n(x) = \exp \left[-\frac{x^2 V(\phi)}{2} \right] \quad \forall x \in \mathbb{R}.$$

For any test function $\phi \in H^s$, define

$$P_\eta(x) = \frac{\eta}{\pi(x^2 + \eta^2)}, \quad \phi_\eta = P_\eta * \phi.$$

At this moment, denote $Z_n(\phi) := Z_n(x)$. Using proposition 1, 2 and the fact that $\|\phi - \phi_\eta\|_s \rightarrow 0$ as $\eta \downarrow 0$, we have

$$\lim_{\eta \downarrow 0} (Z_n(\phi) - Z_n(\phi_\eta)) = 0.$$

Then for any converging subsequence $\{Z_{n_j}(\phi)\}_{j=1}^\infty$ we have

$$\lim_{n_j \rightarrow \infty} Z_{n_j}(\phi) = \lim_{\eta \downarrow 0} \lim_{n_j \rightarrow \infty} Z_{n_j}(\phi_\eta).$$

Proof | Step 2 | The differential equation satisfied by $Z_n(\phi_\eta)$

$$\begin{aligned}\frac{d}{dx}Z_n(\phi_\eta) &= \frac{d}{dx}\mathbb{E}\left[\exp\left(ix\sqrt{\frac{b_n}{n}}\mathcal{N}_n^\circ(\phi_\eta)\right)\right] \\ &= \frac{1}{2\pi}\sqrt{\frac{b_n}{n}}\int_{-\infty}^{\infty}\phi(\mu)(Y_n(z_\mu, x) - Y_n(\bar{z}_\mu, x))d\mu,\end{aligned}$$

where $Y_n(z_\mu, x) = \mathbb{E}[e_{\eta,n}(x)\text{Tr}(G^\circ(z_\mu))]$, $G(z) = \text{Tr}(M - zI)^{-1}$, $e_{\eta,n}(x) = \exp\left(ix\sqrt{\frac{b_n}{n}}\mathcal{N}_n^\circ(\phi_\eta)\right)$, $\mu = \Im(z_\mu)$, and $X^\circ = X - \mathbb{E}[X]$.

$$Y_n(z, x) = Z_n(\phi_\eta) \frac{x}{2\pi} \sqrt{\frac{n}{b_n}} \int_{-\infty}^{\infty} (C_n(z, z_\mu) - C_n(z, \bar{z}_\mu)) \phi(\mu) d\mu,$$

where $C_n(z, z_\mu)$ is written on the next page. Then we have

$$\begin{aligned} \frac{d}{dx} Z_n(\phi_\eta) &= \frac{1}{2\pi} \sqrt{\frac{b_n}{n}} \int_{-\infty}^{\infty} \phi(\mu) (Y_n(z_\mu, x) - Y_n(\bar{z}_\mu, x)) d\mu \\ &= -x Z_n(\phi_\eta) V_n(\phi, \eta), \end{aligned}$$

where

$$\begin{aligned} V_n(\phi, \eta) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\mu_1) \phi(\mu_2) [C_n(z_{\mu_1}, \bar{z}_{\mu_2}) + C_n(\bar{z}_{\mu_1}, z_{\mu_2}) \\ &\quad - C_n(z_{\mu_1}, z_{\mu_2}) - C_n(\bar{z}_{\mu_1}, \bar{z}_{\mu_2})] d\mu_2 d\mu_1 \end{aligned} \quad (5)$$

Proof | Step 4 | Finding the limit of $V_n(\phi, \eta)$

$$\begin{aligned}
 \lim_{n \rightarrow \infty} C_n(z_{\mu_1}, z_{\mu_2}) &= \frac{1}{1 - 2f^2(z_{\mu_1})} \left[f^2(z_{\mu_1})f^2(z_{\mu_2})(1 + 2f'(z_{\mu_2})) \lim_{n \rightarrow \infty} \mathbb{E}[T_n] \right. \\
 &\quad + f^2(z_{\mu_1})f(z_{\mu_2}) \lim_{n \rightarrow \infty} \frac{d}{dz_{\mu_2}} \mathbb{E}[T_n] + \sigma^2 f^2(z_{\mu_1})f^2(z_{\mu_2})(1 + 2f'(z_{\mu_2})) \\
 &\quad \left. + 2\kappa_4 \left\{ f^3(z_{\mu_1})f^3(z_{\mu_2})(1 + 2f'(z_{\mu_2})) + f^3(z_{\mu_1})f(z_{\mu_2})f'(z_{\mu_2}) \right\} \right] (6)
 \end{aligned}$$

where

$$\begin{aligned}
 T_n &= \frac{2}{b_n} \sum_{i,j \in I_1} G_{ij}^{(1)}(z) G_{ij}^{(1)}(z_\mu), \tag{7} \\
 G^{(1)}(z_\mu) &= (M^{(1)} - z_\mu I)^{-1}, \\
 I_1 &= \{1 < i \leq n : d_n(1, i) \leq b_n\},
 \end{aligned}$$

and $M^{(1)}$ is the main bottom $(n - 1) \times (n - 1)$ minor of M .

Proof | Step 4 | Finding the limit of $V_n(\phi, \eta)$

Combining (5), (6), and Proposition 3 we have

$$V(\phi) = \lim_{\eta \downarrow 0} \lim_{n \rightarrow \infty} V_n(\phi, \eta) = \frac{\kappa_4}{16\pi^2} \left(\int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{4 - \mu^2}{\sqrt{8 - \mu^2}} \phi(\mu) d\mu \right)^2 + \frac{\sigma^2}{16\pi^2} \left(\int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\mu \phi(\mu)}{\sqrt{8 - \mu^2}} d\mu \right)^2$$
$$+ \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \sqrt{(8 - x^2)(8 - y^2)} \tilde{F}(x, y) \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\mu_1 \phi(\mu_1)}{(x - \mu_1) \sqrt{8 - \mu_1^2}} \frac{\mu_2 \phi(\mu_2)}{(x - \mu_2)^2 \sqrt{8 - \mu_2^2}} d\mu_1 d\mu_2 dx dy.$$

Proposition 1*

Let M be an $n \times n$ random matrix and $\mathcal{N}_n(\phi)$ be a linear eigenvalue statistic of its eigenvalue as in (3). Then for any $s > 0$ we have

$$\text{Var}[\mathcal{N}_n(\phi)] \leq C_s \|\phi\|_s^2 \int_0^\infty dy e^{-y} y^{2s-1} \int_{-\infty}^\infty \text{Var}[\text{Tr}(G(x + iy))] dx,$$

where C_s is a constant depends only on s , and $G(z) = (M - zI)^{-1}$, is the resolvent of the matrix M .

Proposition 2

Consider symmetric band matrix M defined in (1) and assume (2) is satisfied. Then

$$\text{Var}\{\gamma_n\} \leq \frac{C}{b_n} (|\Im z|^{-2} + |\Im z|^{-4}) \sum_{i=1}^n \mathbb{E}[|G_{ii}|^2] \quad (8)$$

where $\gamma_n = \text{Tr}(M - zI)^{-1} = \text{Tr}(G(z))$.

Proposition 3

Let T_n be as defined in (7). Then

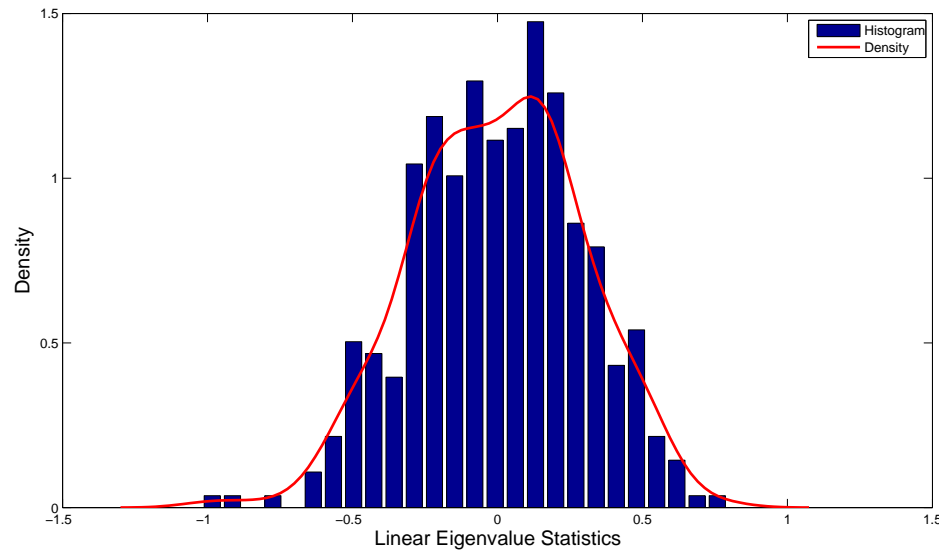
$$\lim_{n \rightarrow \infty} \mathbb{E}[T_n] = \frac{1}{4\pi^3} \int_{-2\sqrt{2}}^{2\sqrt{2}} \int_{-2\sqrt{2}}^{2\sqrt{2}} \frac{\sqrt{8-x^2} \sqrt{8-y^2}}{(x-z)(y-z_\mu)} \tilde{F}(x, y) \mathbf{1}_{\{x \neq y\}} dx dy,$$

where

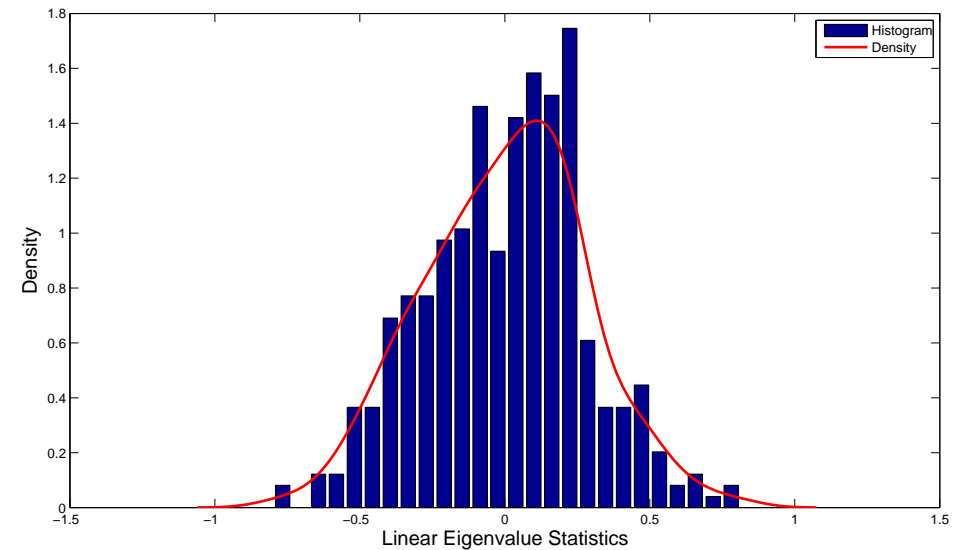
$$\tilde{F}(x, y) = \sqrt{2} \int_{-\infty}^{\infty} \frac{u(u-u^3)}{2(1-u^2) + u^2(x^2+y^2) - u(1+u^2)} ds,$$

and $u = \frac{\sin s}{s}$.

MATLAB Results



(a) $n = 2000$, $b_n = n^{0.2}$. Fourth moment/(variance) $^2=2.92$

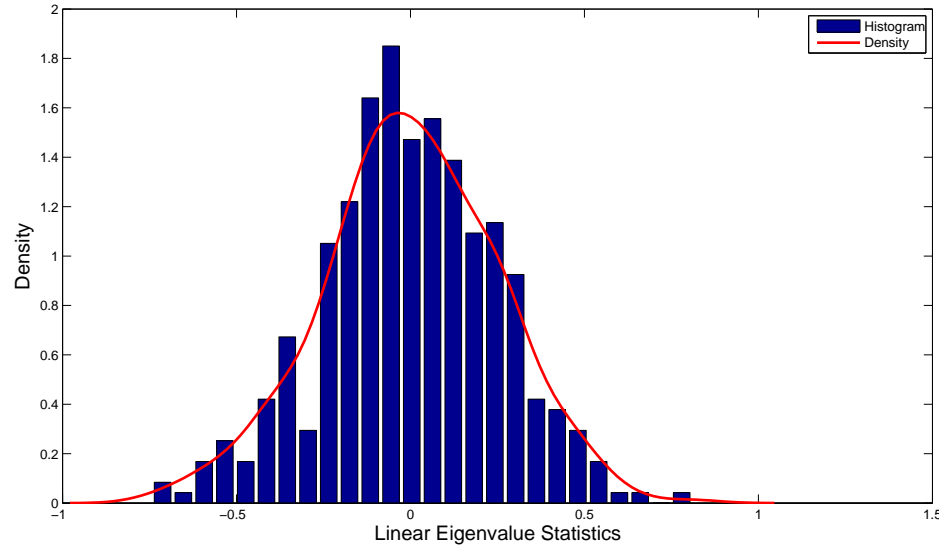


(b) $n = 2000$, $b_n = n^{0.8}$. Fourth moment/(variance) $^2=2.91$

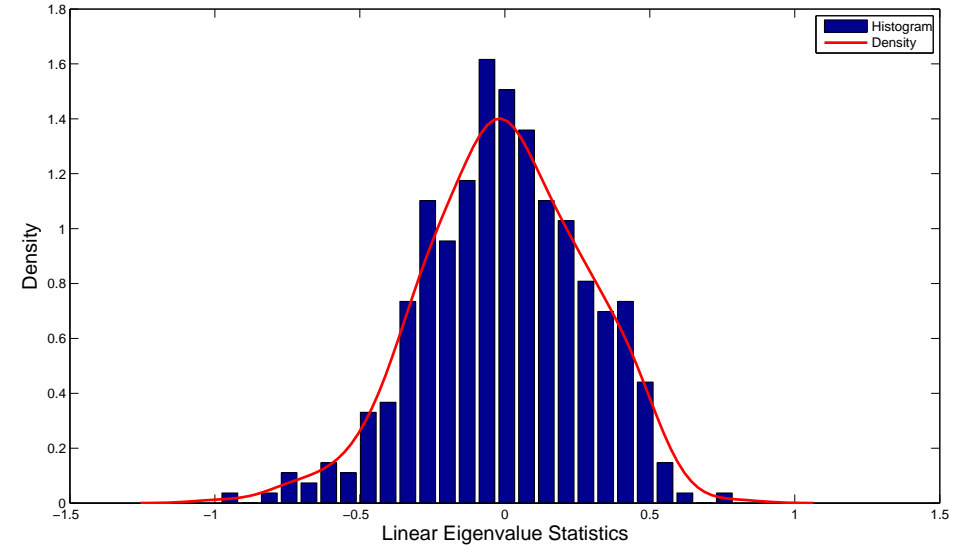
Figure : The eigenvalue statistics was sampled 400 times. The test function was $\phi(x) = \sqrt{16 - x^2}$

MATLAB Results

In the following example we had taken a different test function.



(a) $n = 2000$, $b_n = n^{0.2}$. Fourth moment/(variance) $^2=3.08$



(b) $n = 2000$, $b_n = n^{0.8}$. Fourth moment/(variance) $^2=3.08$

Figure : The eigenvalue statistics was sampled 400 times. The test function was $\phi(x) = e^{-x^2}$

References:

*For the proof of Proposition 1 see the references below

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