University of BRISTOL

# Linear Spectral Statistics for Half-Heavy-Tailed Wigner Matrices

Anna Maltsev, joint work with Florent Benaych-Georges


## 1. Abstract

We consider a Wigner matrix $A$ with entries whose cumulative distribution decays as $x^{-\alpha}$ with $2 < \alpha < 4$ for large $x$. We prove that the fluctuations of the linear statistics $N^{-1}\operatorname{Tr}\varphi(A)$, for some nice test functions $\varphi$, have order $N^{-\alpha/4}$. The behavior of such fluctuations has been understood for both heavy-tailed matrices (i.e. $\alpha < 2$) in [2] and light-tailed matrices (i.e. $\alpha > 4$) in [1]. This paper fills in the gap of understanding for $2 < \alpha < 4$. We find that while linear spectral statistics for heavy-tailed matrices have fluctuations of order $N^{-1/2}$ and those for light-tailed matrices have fluctuations of order $N^{-1}$, the linear spectral statistics for half-heavy-tailed matrices exhibit an intermediate $\alpha$-dependent order of $N^{-\alpha/4}$.


## 2. Wigner Random Matrix

Let $h$ be a probability distribution on $\mathbb{C}$. A **Wigner matrix** $H$ is a $N \times N$ matrix that is:

- Hermitian,
- entries are iid complex *random variables* with distribution $h$,
- mean 0 and variance 1.

We consider $H_N = H/\sqrt{N}$ so that eigenvalues are order 1.

## 3. Our matrices

Let

$$A = [a_{ij}]_{1\le i,j\le N} = \left[\frac{x_{ij}}{\sqrt{N}}\right]_{1\le i,j\le N},$$

where one of two conditions holds: either

(a) $x_{ij}$'s, $1 \le i \le j$, are i.i.d. real random variables with $\mathbb{E}\, x_{ij} = 0$, $\mathbb{E}\,|x_{ij}|^2 = 1$, and (1).

or

(b) $x_{ij} = x_{ij}^R/\sqrt{2} + i x_{ij}^I/\sqrt{2}$ for $1 < i < j$ and $x_{ii} = x_{ii}^R$ where $x_{ij}^I$ and $x_{ij}^R$ are i.i.d. real symmetric random variables $\mathbb{E}\, x_{ij} = 0$, $\mathbb{E}\,|x_{ij}|^2 = 1$, and (1).

## 4. Half-Heavy-Tailed

We say a random variable is half-heavy-tailed if for a certain $\alpha \in (2,4)$ and a certain $c > 0$, as $x \to +\infty$,

$$\mathbb{P}(|x_{ij}| > x) \sim \frac{c}{\Gamma(\alpha+1)} x^{-\alpha}. \qquad (1)$$

Get Wigner semicircle law whenever $\alpha > 2$.

## 5. Our test functions

We prove Gaussian convergence for any random variable of the form

$$\frac{1}{N^{1-\alpha/4}}(\operatorname{Tr}\varphi(A) - \mathbb{E}\operatorname{Tr}\varphi(A)),$$

where $\varphi$ is a function of the type

$$\varphi(\lambda) = \sum_{j=1}^{p} \frac{c_j}{z_j - \lambda}$$

for some $p \ge 1$, $c_1, \ldots, c_p \in \mathbb{C}$, $z_1, \ldots, z_p \in \mathbb{C}\backslash\mathbb{R}$.

## 6. Theorem: Our current work

**Theorem 1.** *For*

$$G(z) := (z - A)^{-1}$$

*with $A$ as above the process*

$$\left(\frac{1}{N^{1-\alpha/4}}(\operatorname{Tr} G(z) - \mathbb{E}\operatorname{Tr} G(z))\right)_{z\in\mathbb{C}\backslash\mathbb{R}}$$

*converges to a complex Gaussian centered process $(X_z)_{z\in\mathbb{C}\backslash\mathbb{R}}$ with covariance defined by the fact that $X_{\overline{z}} = \overline{X_z}$ and that for any $z, z' \in \mathbb{C}\backslash\mathbb{R}$, $\mathbb{E}[X_z X_{z'}] = C(z,z')$, for*

$$C(z,z') := -\iint_{t,t'>0} \partial_z \partial_{z'} \Big\{ [(K(z,t) + K(z',t'))^{\alpha/2} - (K(z,t)^{\alpha/2} + K(z',t')^{\alpha/2})] \exp\left(\operatorname{sgn}_z itz - K(z,t) + \operatorname{sgn}_{z'} it'z' - K(z',t')\right) \Big\} \frac{c\,\mathrm{d}t\,\mathrm{d}t'}{2tt'}$$

*where $c$ and $\alpha$ are as in* (1), $\operatorname{sgn}_z := \operatorname{sgn}(\Im z)$ *and* $K(z,t) := \operatorname{sgn}_z itG_{\mathrm{sc}}(z)$, $G_{\mathrm{sc}}(z)$ *being the Stieltjes transform of the semicircle law with support* $[-2,2]$.

## 7. Proof Outline

1. Truncate, renormalize, and, in the real case, centralize
2. Use the Martingale decomposition and the Central Limit Theorem for Martingales
3. Show that the off-diagonal terms of the resolvent do not contribute to the limit
4. Compute the limit with just the diagonal terms

## 8. Some Details

1. To truncate: We use $|\operatorname{Tr}(G_B(z) - G_A(z))| \le 2|\Im z|^{-1}\operatorname{rank}(B - A)$. Let $B = [a_{ij}\mathbb{1}_{|x_{ij}|\le N^\beta} - \mu_N/\sqrt{N}]$. We will solve for $\beta$. Subtracting $\mu_N/\sqrt{N}$ from each matrix entry is a rank 1 perturbation. Then, as $\mathbb{P}(|x_{ij}| > N^\beta) \le CN^{-\alpha\beta}$, we have $\operatorname{rank}(B - A) \le 1 + 2\sum_{i=1}^N X_i$ where the $X_i$'s are independent Bernoulli r.v. with parameters $\mathbb{P}(X_i = 1) = 1 - (1 - CN^{-\alpha\beta})^i$. Need $(2 - \alpha\beta)_+ < 1 - \alpha/4$, *i.e.* $\beta > \frac{2-(1-\alpha/4)}{\alpha} = \frac{1}{4}(1 + \frac{\alpha}{4})$.

2. Martingale: We will use a CLT for Martingales for $M(\varphi, N)$, with $M_N(N) = M(\varphi, N)$ and $\mathcal{F}_k(N) := \sigma(x_{i,j}\ ;\ i \le k \text{ and } j \le k)$.

   Then, denoting $\mathbb{E}[\,\cdot\,|\mathcal{F}_k]$ by $\mathbb{E}_k$, the random variable $Y_k(N)$ of CLT for Martingales is

   $$Y_k = Y_k(N) = \frac{1}{N^{1-\alpha/4}}(\mathbb{E}_k - \mathbb{E}_{k-1})(\operatorname{Tr}\varphi(A)).$$

   We use $\varphi = \frac{1}{z-x}$

3. Rewrite $Y_k$'s as a log derivative; then by Cauchy Inequality suffices to bound the log

   $$Y_k = \frac{1}{N^{1-\alpha/4}}(\mathbb{E}_k - \mathbb{E}_{k-1})\frac{1 + \mathbf{a}_k^*(G^{(k)}(z))^2\mathbf{a}_k}{z - a_{kk} - \mathbf{a}_k^* G^{(k)}(z)\mathbf{a}_k}$$
   $$= \partial_z \log|z - a_{kk} - \mathbf{a}_k^* G^{(k)}(z)\mathbf{a}_k|^2$$

4. Remove off-diagonal terms: we define define

   $$\tilde{Y}_k := \frac{1}{N^{1-\alpha/4}}(\mathbb{E}_k - \mathbb{E}_{k-1})\frac{1 + \mathbf{a}_k^*(G^{(k)}(z))^2_{\mathrm{diag}}\mathbf{a}_k}{z - \mathbf{a}_k^* G^{(k)}(z)_{\mathrm{diag}}\mathbf{a}_k}$$

   and $\tilde{Y}_k'$ with $z'$ instead of $z$. Then $\sum_{k=1}^N \mathbb{E}_{k-1}[Y_k Y_k'] - \mathbb{E}_{k-1}[\tilde{Y}_k \tilde{Y}_k']$ converges in probability to 0.

5. To compute the limit: For $w \in \mathbb{C}\backslash\mathbb{R}$, $\frac{1}{w} = -i\operatorname{sgn}_w \times \int_0^{+\infty} e^{\operatorname{sgn}_w itw}\mathrm{d}t$, with $\operatorname{sgn}_w = \operatorname{sgn}(\Im w)$ then $\frac{1+\sum_j |\mathbf{a}_k(j)|^2(G^{(k)}(z))^2_{jj}}{z-\sum_j |\mathbf{a}_k(j)|^2 G^{(k)}(z)_{jj}} = -\int_0^{+\infty}\frac{1}{t}\partial_z\{e^{\operatorname{sgn}_z it(z-\sum_j |\mathbf{a}_k(j)|^2 G^{(k)}(z)_{jj})}\}\mathrm{d}t$. Then taking $\mathbb{E}_k$ the exponent splits into a sum of independent r.v.'s, and the integral into a product of Laplace transforms.

## 9. Remarks

1. If $\alpha > 4$, the eigenvalues of $A$ fluctuate very little. The heavier the tails the more similar to sparse $A$ is, and the more independently its eigenvalues behave. When $\alpha < 2$, the order of fluctuations is same as for independent r.v.'s. Half-heavy-tailed matrices show an exciting transitional regime (cf TABLE).
2. Our test functions $\varphi$ of this type span (by closure) the set of continuous functions $\to 0$ at $\pm\infty$ by the Stone-Weierstrass theorem.
3. For real symmetric case, subtracting the mean from each matrix entry after truncation is rank 1, but it's not rank 1 for complex hermitian, hence need symmetric r.v.'s.


## References

[1] Z. D. Bai, J. W. Silverstein *Spectral analysis of large dimensional random matrices*, Second Edition, Springer, New York, 2009.

[2] F. Benaych-Georges, A. Guionnet, C. Male *Central limit theorems for linear statistics of heavy tailed random matrices*. Comm. Math. Phys.Vol. 329 (2014), no. 2, 641–686.

[3] F. Benaych-Georges, S. Péché *Localization and delocalization for heavy tailed band matrices*, Ann. Inst. Henri Poincaré Probab. Stat., Vol. 50 (2014), no. 4, 1385–1403.


## TABLE

Orders of the fluctuations of the r.v. $\frac{1}{N}\operatorname{Tr} G$ of around its expectation as a function of the exponent $\alpha$ st $\mathbb{P}(|a_{ij}| > x) \approx x^{-\alpha}$ for $x$ large.

| | $\alpha < 2$ | $2 < \alpha < 4$ | $\alpha > 4$ |
|---|---|---|---|
| Order of the fluctuations | $N^{-1/2}$ | $N^{-\alpha/4}$ | $N^{-1}$ |