Jacobi Polynomial Moments and Products of Random Matrices

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Products of Random Matrices

- We consider products of independent matrices of the form

$$\boxed{Y_{r,s} = G_r \dots G_{s+1} T_s \dots T_1}$$

where $T_1, \dots, T_s$ are truncations of Haar distributed unitary matrices and $G_{s+1}, \dots, G_r$ are complex Ginibre random matrices ($s < r$).
- Here, the j-th matrix has dimension $(n + \nu_j) \times (n + \nu_{j-1})$, where $\nu_j \geq 0$, $\nu_0 = 0$, and each T_j can be considered as the left upper block of a Haar distributed unitary random matrix of size $l_j \geq 2n + \nu_j + \nu_{j-1}$ for $j = 1, \dots, r$.
- We look at the Wishart-type matrices of size $n \times n$

$$\boxed{Z_{r,s} = Y_{r,s}^* Y_{r,s}}$$

Random matrices of this type arise in mathematical physics and applications, e.g., in Multiple-Input and Multiple-Output communication (MIMO) [2].
- We are interested in the *global regime* of $Z_{r,s}$, i.e. the limiting distribution of eigenvalues of $Z_{r,s}$ as $n \to \infty$ after suitable rescaling.

Known Facts and Results

- For finite dimensions n the eigenvalues of $Z_{r,s}$ form a determinantal point process with joint probability density function involving Meijer G-functions [1], [4].
- If all ν_j and $l_j - 2n$ remain fixed as $n \to \infty$, then the empirical eigenvalue distributions of $\frac{1}{n^{r-s}} Z_{r,s}$ converge weakly almost surely to the *free multiplicative convolution* (as introduced by Voiculescu)

$$R_{r-s+1,1} \boxtimes R_{1,\frac{1}{2}}^{\boxtimes s}.$$

- Here, for $\alpha, \beta \in \mathbb{R}$, $\alpha \geq 1$ and $0 \leq \beta \leq \alpha$ the *Raney distribution* $R_{\alpha,\beta}$ is a compactly supported measure on $[0, \infty)$ defined by the moment sequence [3], [5]

$$R_{\alpha,\beta}(n) = \frac{\beta}{n\alpha + \beta} \binom{n\alpha + \beta}{n}.$$

The *Fuss-Catalan distributions* and the *arcsine measure* transformed to $[0,1]$ belong to the class of Raney distributions.
- It turns out that only in the special cases $s = 0$ and $s = 1$ the global regimes are contained in the Raney class. We have, for instance,

$$R_{r,1} \boxtimes R_{1,\frac{1}{2}} = R_{\frac{r+1}{2},\frac{1}{2}}.$$

Question

Can we find an explicit characterization for the global regimes

$$R_{r-s+1,1} \boxtimes R_{1,\frac{1}{2}}^{\boxtimes s}$$

by means of their moments?

Jacobi Polynomial Moments

For $a > 0$ and $r, s \in \mathbb{N} \cup \{0\}$ such that $s < r$ we define the sequence $J_{r,s,a}(n)$ by

$$J_{r,s,a}(0) = a$$

and for $n \geq 1$

$$\boxed{J_{r,s,a}(n) = \frac{a}{n} \left(\frac{a^r}{(1+a)^s} \right)^n P_{n-1}^{(\alpha_{n-1},\beta_{n-1})} \left(\frac{1-a}{1+a} \right)}$$

where $P_n^{(\alpha_n,\beta_n)}(x)$ are the *Jacobi polynomials with varying parameters* $\alpha_n = rn + r + 1$ and $\beta_n = -(r + 1 - s)n - (r + 2 - s)$.

Moreover, we define the quantities

$$w^* = \frac{a(r+1-s) - r + \sqrt{(a(r+1-s) - r)^2 + 4a(r+1)(r-s)}}{2(r-s)},$$

$$x^* = \frac{r+1}{s+1} \frac{(w^*)^r}{(w^*+1)^{s-1} \left(w^* - \frac{as-1}{s+1} \right)},$$

and

$$\tilde{w} = \frac{a(r+1-s) - r - \sqrt{(a(r+1-s) - r)^2 + 4a(r+1)(r-s)}}{2(r-s)},$$

$$\tilde{x} = \frac{r+1}{s+1} \frac{(\tilde{w})^r}{(\tilde{w}+1)^{s-1} \left(\tilde{w} - \frac{as-1}{s+1} \right)}.$$

Main Result: Solving a Moment Problem

Let $r, s \in \mathbb{N} \cup \{0\}$ such that $s < r$ and let a be a positive real number. Then there exists a unique measure $J_{r,s,a}$ on $[0, x^*]$ with total mass a such that the moments are given by the Jacobi polynomial sequence $(J_{r,s,a}(n))$.

A Riemann surface

The solution of the moment problem is based on a study of the Riemann surface associated with the algebraic equation

$$w^{r+1} - x(w-a)(w+1)^s = 0.$$

The equation defines an algebraic function on a compact surface of genus 0 with branch points at 0, x^* and $\tilde{x}$. This function can be used to construct the solution.

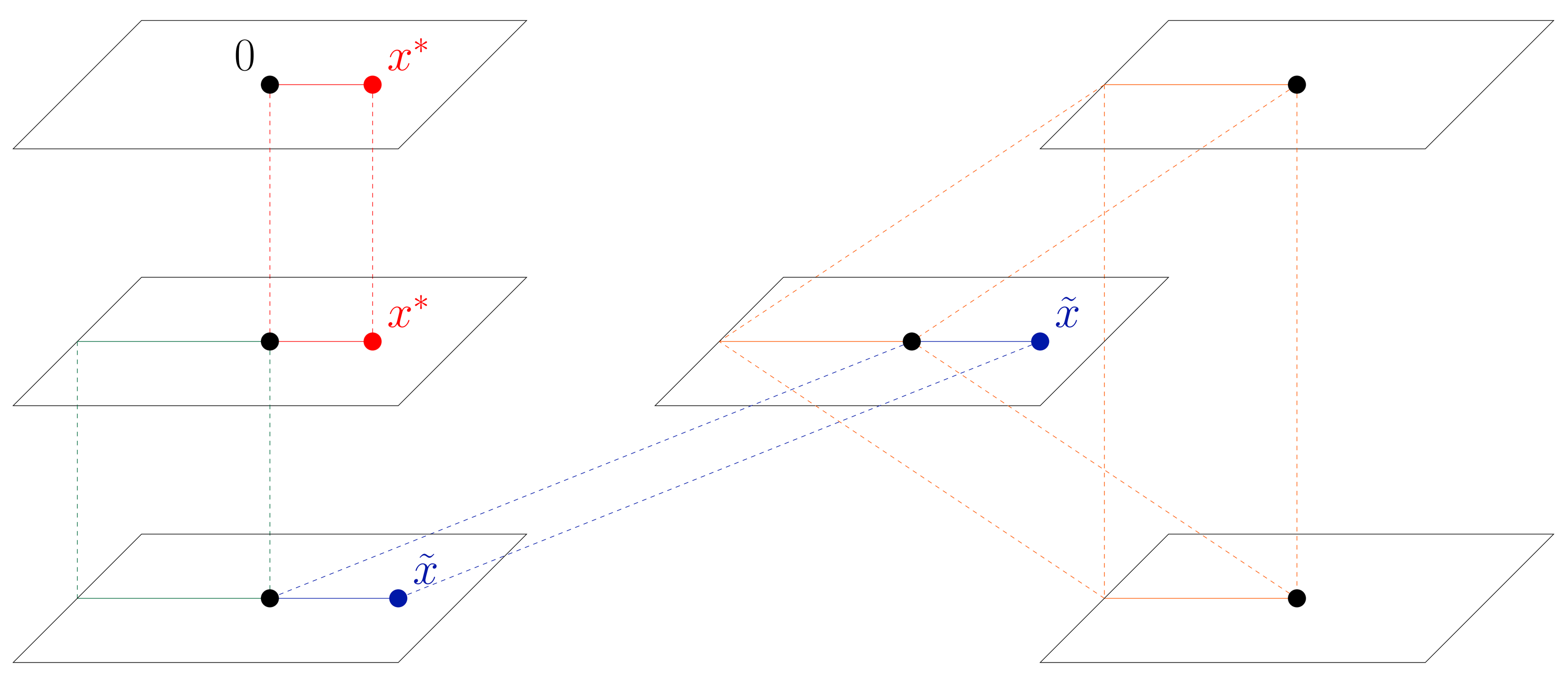

Figure 1: The geometry of the Riemann surface in the case $r = 5, s = 3$.

It turns out that we can give an affirmative answer to the question raised above.

Answer to the Question

The global regimes are contained in the class of measures given by the Jacobi Polynomial moments. More precisely, we have

$$R_{r-s+1,1} \boxtimes R_{1,\frac{1}{2}}^{\boxtimes s} = J_{r,s,1}.$$

As a consequence from the theory of Jacobi polynomials we obtain explicit representations for the moments of the global regimes.

Some open Questions

- Explicit representations for the densities of the measures $J_{r,s,a}$ are known only in the special cases $s = 0$ and $s = 1$. Is it possible to describe the densities in more general cases?
- We restricted the above analysis to $s < r$. What happens in the case $s = r$ in which the product only consists of truncated Haar distributed unitary matrices?

Some References

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