# Wronskians, Duality and Cardy Branes

Hirotaka Irie[1] and Benjamin Niedner[2]

[1]Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan
[2]Rudolf Peierls Institute for Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, UK
E-Mail: irie@yukawa.kyoto-u.ac.jp, benjamin.niedner@physics.ox.ac.uk

## Abstract

We calculate expectations of products of characteristic polynomials in the double-scaled hermitian two-matrix ensemble. When all spectral parameters coincide, these arise as solutions to an extension of the reduced KP hierarchy and are naturally expressed as a generalised Wronskian. The solutions are in one-to-one correspondence with entries in the Kac table of a conformal minimal model and we derive a Lax system that associates a spectral curve to each entry. We show that in the weak-coupling limit, our results reduce to the decomposition of Cardy branes into principal FZZT branes conjectured by Seiberg and Shih in the context of minimal string theory.

## Hermitian two-matrix ensemble

We consider expectations of products of characteristic polynomials

$$\alpha_n^{(M)}(x_1,\dots x_M)=\left\langle \prod_{i=1}^{M}\det(x_i-X)\right\rangle_{n\times n},$$

where the average is taken in the hermitian two-matrix ensemble

$$\langle\mathcal{O}\rangle_{n\times n}=\frac{1}{Z}\int_{1\le n\le N}\mathrm{d}X\mathrm{d}Y\,e^{-N\mathrm{tr}[V_1(X)+V_2(Y)-XY]}\mathcal{O},$$

with suitably chosen polynomials $\deg V_1=p$, $\deg V_2=q$. In the double scaling limit, $\alpha_n^{(M)}$ computes the partition function of $M$ branes in minimal string theory with target space positions $x_i$ [1]. At finite $N$, the exact, anti-symmetrised expression in terms of orthogonal polynomials $\{\alpha_n^{(1)}\equiv\alpha_n\}_{n=1}^N$ reads [2]

$$\alpha_n^{(M)}(x_1,\dots x_M)=\frac{\det_{1\le a,b\le M}\alpha_{n+1-a}(x_b)}{\det_{1\le a,b\le M}x_a^{b-1}}.$$

Setting $x_i=x_c+\varepsilon\zeta_i$, $n/N=\exp(-\varepsilon^\mu t)$ and taking the double scaling limit $N=1/g\to\infty$, $\varepsilon\to 0$, the RHS becomes [1]

$$\frac{\det_{1\le a,b\le M}\left(\partial_t^{a-1}\psi(t;\zeta_b)\right)}{\det_{1\le a,b\le M}\zeta_a^{b-1}},\quad \partial_t=g\frac{\partial}{\partial t},\qquad (1)$$

where the wave function $\psi$ of the so-called principal FZZT brane solves the $p^{\text{th}}$ and $q^{\text{th}}$-order differential equations

$$\zeta\psi(t;\zeta)=\mathbb{P}(t;\partial_t)\psi(t;\zeta),\quad \partial_\zeta\psi(t;\zeta)=\mathbb{Q}(t;\partial_t)\psi(t;\zeta),\qquad (2)$$

with compatibility condition $[\mathbb{P},\mathbb{Q}]=g$ ($p^{\text{th}}$ reduction of KP hierarchy [2, 3]) and

$$\mathbb{P}(t;\partial_t)=2^{p-1}g\partial_t^p+\sum_{n=2}^{p}u_n(t)\partial_t^{p-n},$$

$$\mathbb{Q}(t;\partial_t)=2^{q-1}g\partial_t^q+\sum_{n=2}^{q}v_n(t)\partial_t^{q-n}.$$

Generally, the fermionic statistics prevent us from taking a coincidence limit of multiple FZZT branes $\zeta_i\to\zeta_j$, $i\neq j$ in (1). However, a nontrivial structure emerges when combining the $p$ independent solutions to (2), whose label serves as an additional quantum number. The analogous notions can be defined for the matrix $Y$ in terms of the dual system as

$$\partial_\eta\chi(t;\zeta)=\mathbb{P}(t;\partial_t)^T\chi(t;\zeta),\quad \eta\chi(t;\zeta)=\mathbb{Q}(t;\partial_t)^T\chi(t;\zeta),\qquad (3)$$

where $(f(t)\circ\partial_t^n)^T=(-\partial_t)^n\circ f(t)$.

## Generalised Wronskian I

Let $\{\psi^{(j)}\}_{j=1}^p$ denote the set of $p$ independent solutions to (2). Given a partition $\lambda$, we define the generalised Wronskian

$$W_\lambda^{(n)}(t;\zeta)=\det_{1\le a,b\le n}\left(\partial_t^{a-1+\lambda_a}\psi^{(j_b)}(t;\zeta_b)\right)\Big|_{\zeta_1=\zeta_2=\dots\zeta}$$

which is nontrivial for $1\le n\le p-1$. This may be expressed in terms of the Schur polynomial $S_\lambda$

$$W_\lambda^{(n)}(t;\zeta)=S_\lambda(\{\partial_k\}_{k=1}^n)W_\varnothing^{(n)}(t;\zeta),$$

where $\varnothing$ denotes the diagram with $\lambda_a=0\ \forall a$ and $\partial_k\psi^{(j_b)}=\delta_{bk}\partial_t\psi^{(j_b)}$. From the defining equations, we obtain $p^{\text{th}}$ and $q^{\text{th}}$-order differential operators such that

$$\zeta W_\lambda^{(n)}(t;\zeta)=\mathbb{P}_\lambda^{(n)}(t;\partial_t)W_\varnothing^{(n)}(t;\zeta),\quad \partial_\zeta W_\lambda^{(n)}(t;\zeta)=\mathbb{Q}_\lambda^{(n)}(t;\partial_t)W_\varnothing^{(n)}(t;\zeta).$$

The $\binom{p}{n}$ independent solutions to the above equations form the $\binom{p}{n}\times\binom{p}{n}$ linear system

$$\partial_t\vec{W}^{(n)}(t;\zeta)=\mathcal{B}^{(n)}(t;\zeta)\vec{W}^{(n)}(t;\zeta),\quad \partial_\zeta\vec{W}^{(n)}(t;\zeta)=\mathcal{Q}^{(n)}(t;\zeta)\vec{W}^{(n)}(t;\zeta).$$

E.g. when $p=4$, $n=2$ a basis is

$$\vec{W}^{(2)}=\left(W_\varnothing^{(2)},W_{\square}^{(2)},W_{\square\square}^{(2)},W_{\boxminus}^{(2)},W_{\square\square,\square}^{(2)},W_{\boxplus}^{(2)}\right).$$

The spectrum of the Lax pair $\mathcal{B}^{(n)}$, $\mathcal{Q}^{(n)}$ can be retrieved from the polynomials

$$F^{(n)}(t;P,z)=\det\left(\mathbb{1}_{\binom{p}{n}}z-\mathcal{B}^{(n)}(t;P)\right),$$

$$G^{(n)}(t;P,Q)=\det\left(\mathbb{1}_{\binom{p}{n}}Q-\mathcal{Q}^{(n)}(t;P)\right).$$

We define the spectral curve of the system by

$$\mathcal{C}_{p,q}^{(n)}(t)=\{(P,Q)\in\mathbb{C}^2|G^{(n)}(t;P,Q)=0\},$$

which is uniformised by the solutions $P(z)$ to $G^{(n)}(t;P,z)=0$.

## Generalised Wronskian II

Work in progress with Chuan-Tsung Chan (Tunghai U.) and Chi-Hsien Yeh (NCTS). From the $q$ solutions $\{\chi^{(j)}\}_{j=1}^q$ of the dual system (3), we define

$$\tilde{W}_\lambda^{(m)}(t;\eta)=\det_{1\le a,b\le m}\left(\partial_t^{a-1+\lambda_a}\chi^{(j_b)}(t;\eta)\right),$$

which is nontrivial for $1\le m\le q-1$. The solution set and its dual are related by a simple Laplace transform:

$$\psi(t;\zeta)=\mathcal{L}[\tilde{W}^{(q-1)}](\zeta),\quad \chi(t;\eta)=\mathcal{L}[W^{(p-1)}](\eta)$$

This leads us to a general proposal for $(p-1)(q-1)$ functions, with $1\le n\le p-1$ and $1\le m\le q-1$

$$\mathcal{W}_{\lambda'\lambda}^{(n|m)}(t;\zeta)=\mathcal{L}_{\lambda'}^{\otimes n}\left[\tilde{W}_\lambda^{(n(q-m))}\right](t;\zeta_1,\dots\zeta_n)\Big|_{\zeta_1=\zeta_2=\dots\zeta}$$

with a generalised Laplace transform $\mathcal{L}_\lambda^{\otimes n}$ on $n$ variables. These are in one-to-one correspondence with entries of the Kac table for the $(p,q)$ Virasoro minimal model, cf. Table 1.

| $\mathcal{L}[\chi]$ | $W^{(r)}[\mathcal{L}[\chi]]$ | $W^{(p-1)}[\mathcal{L}[\chi]]$ |
|---|---|---|
| $\mathcal{L}[\tilde{W}^{(q-s)}]$ | $\mathcal{W}^{(r\|s)}$ | $\mathcal{W}^{(p-1\|s)}$ |
| $\psi$ | $W^{(r)}$ | $W^{(p-1)}$ |

Table 1: One-to-one correspondence between the generalised Wronskians and entries in the Kac table of the $(p,q)$-model

## Main Results

We considered expectations of products of characteristic polynomials in the hermitian two-matrix ensemble with coinciding spectral parameters. We

1. expressed the averages as a generalised Wronskian;
2. found the associated differential equations and $\binom{p}{n}\times\binom{p}{n}$ Lax system;
3. generalised the observations of [4] to the latter;
4. defined the quantum spectral curve and showed that as $g\to 0$, the solutions realise the "Cardy branes" of minimal string theory non-perturbatively;
5. found evidence for a Kac table of $(p-1)(q-1)/2$ allowed observables that describe the full brane spectrum, using duality.

## Semiclassical limit

Our construction achieves the long-missing realisation of the remaining of the $(p-1)(q-1)/2$ branes (one for each Cardy state of the minimal model, which describes the "matter sector" of minimal string theory) in the matrix model: consider solutions to $[\mathbb{P},\mathbb{Q}]=g$ that admit a power series expansion in $g=1/N$

$$u_n(t)=\sum_{k\ge 0}g^k u_n^{[k]}(t),\quad v_n(t)=\sum_{k\ge 0}g^k v_n^{[k]}(t),$$

such that $u_n^{[0]}(t)$, $v_n^{[0]}(t)$ are constant ("Liouville background", cf. e.g. [3, 5]). Then to leading order as $g\to 0$, the principal FZZT brane $\mathcal{W}_\varnothing^{(1|1)}=\psi$ is described by the spectral curve [6]

$$0=T_p(Q)-T_q(P).$$

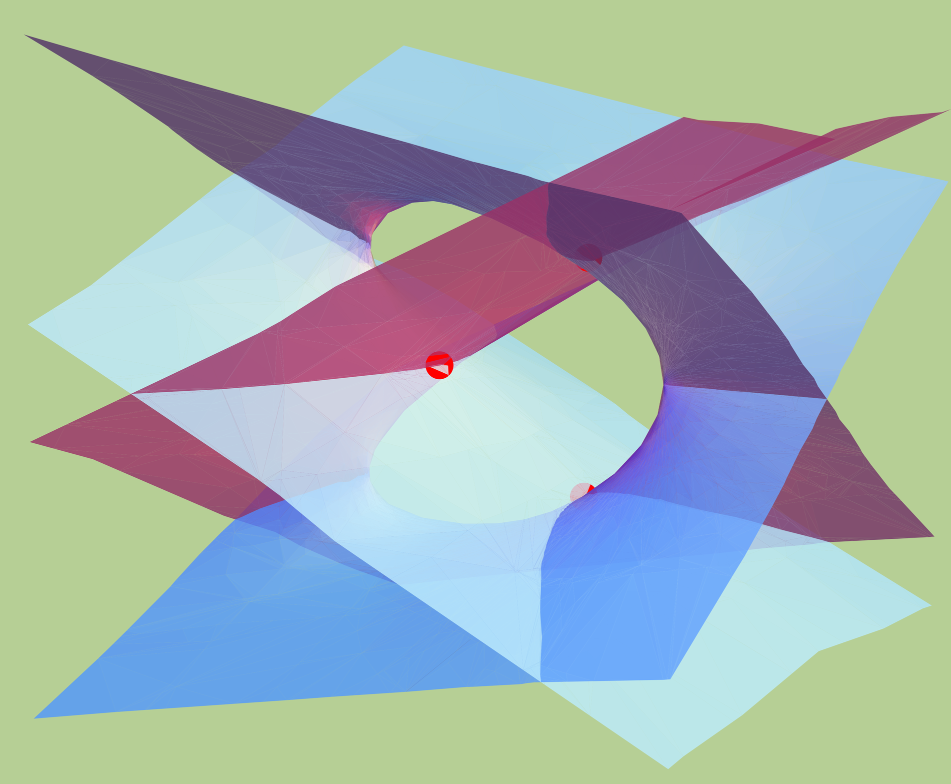

Figure 1: Semiclassical curve for $(p,q)=(3,4)$

In the limit $g\to 0$, the spectral curve $\mathcal{C}_{p,q}^{(n)}(t)$ then factorises into disconnected pieces. E.g. when $n=2$, the semiclassical spectral curve is given by $(P,Q)\in\mathbb{C}^2$ s.t.

$$0=\prod_{k=1}^{(p-1)/2}\left[T_p(Q/d_k)-T_q(P)\right]$$

for odd $p$ and

$$0=Q^{p/2}\prod_{k=1}^{(p-2)/2}\left[T_p(Q/d_k)-T_q((-1)^kP)\right]$$

for even $p$, with

$$d_k=2\cos(2\pi qk/p),\quad T_p(\cos\phi)=\cos p\phi.$$

The factors $d_k$ are precisely the boundary entropies associated with non-trivial Cardy states of the minimal model [6].

[1] J. M. Maldacena, G. W. Moore, N. Seiberg, and D. Shih, "Exact vs. semiclassical target space of the minimal string," *JHEP*, vol. 0410, p. 020, 2004.
[2] A. Morozov, "Integrability and matrix models," *Phys.Usp.*, vol. 37, pp. 1–55, 1994.
[3] M. Bergère, G. Borot, and B. Eynard, "Rational differential systems, loop equations, and application to the q-th reductions of KP," *ArXiv e-prints*, Dec. 2013.
[4] M. Bertola, B. Eynard, and J. P. Harnad, "Duality, biorthogonal polynomials and multimatrix models," *Commun.Math.Phys.*, vol. 229, pp. 73–120, 2002.
[5] C.-T. Chan, H. Irie, and C.-H. Yeh, "Duality Constraints on String Theory: Instantons and spectral networks," 2013.
[6] N. Seiberg and D. Shih, "Branes, rings and matrix models in minimal (super)string theory," *JHEP*, vol. 0402, p. 021, 2004.