First gap at the edge of the spectrum of Hermitian random matrices

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Abstract

We study the phenomenon of "crowding" near the largest eigenvalue $\lambda_{\max}$ of random $N \times N$ matrices belonging to the Gaussian Unitary Ensemble (GUE) of random matrix theory [1]. We focus on the probability density function of the gap between the first two largest eigenvalues, $p_{\rm GAP}(r,N)$. In the edge scaling limit where $r = \mathcal{O}(N^{-1/6})$, which is described by a double scaling limit of a system of semi-classical orthogonal polynomials, we show that $p_{\rm GAP}(r,N)$ is characterized by a scaling functions which can be expressed in terms of the solution of a Lax pair associated to the Painlevé XXXIV equation. Our expression allows us to obtain precise asymptotic behaviors of the scaling functions both for small and large arguments.

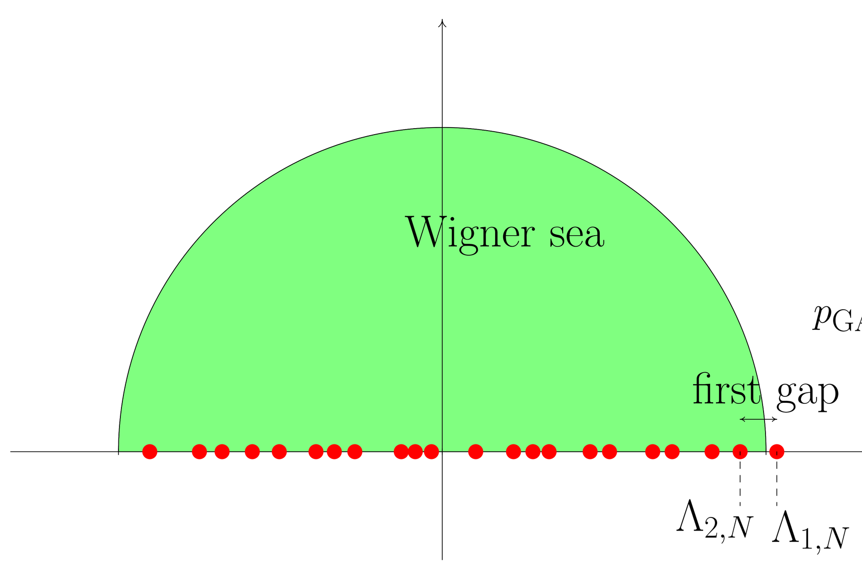
1 First gap of Hermitian random matrices

We consider random Hermitian matrices, of size $N \times N$, belonging to the Gaussian Unitary Ensemble (GUE). The joint PDF of the eigenvalues is given by

$$P_{\rm joint}(\lambda_1, \lambda_2, ..., \lambda_N) = \frac{1}{Z_N} \prod_{i<j} (\lambda_i - \lambda_j)^2 \exp\left(-\sum_{i=1}^N \lambda_i^2\right) ,$$

where Z_N is the normalization constant.

✎ We introduce the first gap between the two largest eigenvalues characterizing the "crowding" near the largest eigenvalue $\lambda_{\max}$. If one denotes by $\lambda_{\max} = \Lambda_{1,N} \geq \Lambda_{2,N} \geq \cdots \geq \Lambda_{N,N}$ we compute the PDF $p_{\rm GAP}(r,N)$ of the spacing between the two largest eigenvalues $\Lambda_{1,N} - \Lambda_{2,N}$:



$$p_{\rm GAP}(r,N){\rm d}r = \Pr.[(\Lambda_{1,N} - \Lambda_{2,N}) \in [r, r+{\rm d}r]] .$$

2 Exact formula for finite N

Using the invariance under any permutation of $P_{\rm joint}(\lambda_1, \lambda_2, ..., \lambda_N)$, $p_{\rm GAP}(r,N)$ can be written as

$$p_{\rm GAP}(r,N) = N(N-1) \int_{-\infty}^{+\infty} \overbrace{{\rm d}y}^{\text{actual value of } \Lambda_{2,N}} \underbrace{\int_{-\infty}^{y} {\rm d}\lambda_1 \int_{-\infty}^{y} {\rm d}\lambda_2 \cdots \int_{-\infty}^{y} {\rm d}\lambda_{N-2}}_{\text{the } N-2 \text{ other eigenvalues are smaller than } \Lambda_{2,N}} P_{\rm joint}(\lambda_1, \cdots, \lambda_{N-2}, \overbrace{y, \underbrace{y+r}_{\Lambda_{1,N}}}^{\text{the first gap is } r}) .$$

We introduce the semi-classical family of monic orthogonal polynomials $\{\pi_k(\lambda,y)\}_{k\in\mathbb{N}}$ wich are polynomials of degree k of the variable λ while y is a parameter [2]:

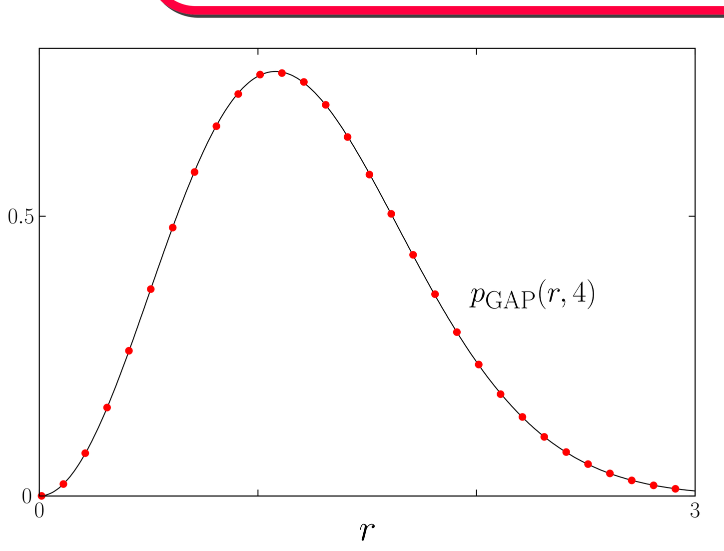
$$\begin{cases} \langle \pi_k, \pi_{k'} \rangle = \int_{-\infty}^{y} {\rm d}\lambda \, \pi_k(\lambda,y)\pi_{k'}(\lambda,y)e^{-\lambda^2} = \delta_{k,k'} \, h_k(y) \\ \pi_k(\lambda,y) = \lambda^k + ... \end{cases}$$

We can express K_N the kernel associated to the orthogonal polynomials and F_N the cumulative distribution of the largest eigenvalue

$$K_N(\lambda,\lambda') = \sum_{k=0}^{N-1} \frac{1}{h_k(y)} \pi_k(\lambda,y)\pi_k(\lambda',y)e^{-\frac{\lambda^2+\lambda'^2}{2}} , \qquad F_N(y) = \Pr.(\lambda_{\max} \leq y) = \frac{N!}{Z_N} \prod_{j=0}^{N-1} h_j(y) .$$

We find after some calculation

$$p_{\rm GAP}(r,N) = \int_{-\infty}^{\infty} {\rm d}y \left[F_N'(y) K_N(y+r, y+r) - F_N(y) K_N^2(y, y+r)\right] .$$



Comparison between numerical result obtained for a 4×4 GUE matrix (red dots) and analytical results (black line).

3 Large N asymptotics

We focus on the *typical* fluctuations of the gap for large N, when $r = \mathcal{O}(N^{-1/6})$

$$p_{\rm GAP}(r,N) = \sqrt{2} N^{1/6} \tilde{p}_{\rm typ}\left(r\sqrt{2}N^{1/6}\right) ,$$

with $\tilde{p}_{\rm typ}(\tilde{r})$ given by

$$\tilde{p}_{\rm typ}(\tilde{r}) = \frac{2^{1/3}}{\pi} \int_{-\infty}^{\infty} {\rm d}x \left[\tilde{f}^2(-\tilde{r},x) - \left(\int_x^{\infty} q(u)\tilde{f}(-\tilde{r},u){\rm d}u\right)^2\right] \mathcal{F}_2(x) ,$$

where $\mathcal{F}_2(x)$ is the Tracy-Widom distribution associated to GUE and $q(x)$ is the Hastings-McLeod solution of Painlevé II with parameter $\alpha = 0$:

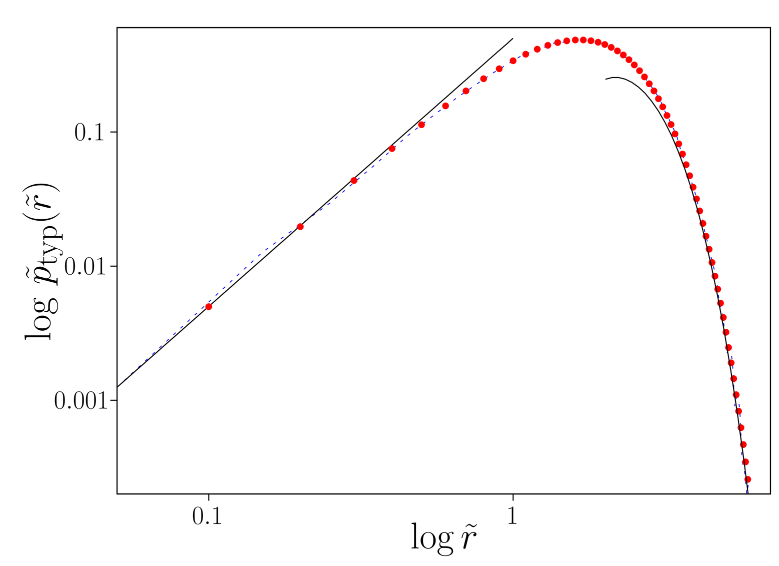
$$\mathcal{F}_2(x) = \exp\left[-\int_x^{\infty} (u-x)q^2(u){\rm d}u\right] , \qquad \begin{cases} q''(x) = 2q^3(x) + x\,q(x) , \\ q(x) \sim {\rm Ai}(x) \text{ for } x \to \infty . \end{cases}$$

The function $\tilde{f}(\tilde{r},x)$ satisfies a Schrödinger like equation with a prescribed asymptotic behavior

$$\partial_x^2 \tilde{f}(\tilde{r},x) - [x + 2q^2(x)]\tilde{f}(\tilde{r},x) = -\tilde{r}\tilde{f}(\tilde{r},x) , \qquad \tilde{f}(\tilde{r},x) \underset{x\to\infty}{\sim} 2^{-1/6}\sqrt{\pi}{\rm Ai}(x-\tilde{r}) .$$

Our expression allows us to find the asymptotic behaviors

$$\tilde{p}_{\rm typ}(\tilde{r}) = \begin{cases} \frac{1}{2}\tilde{r}^2 + \mathcal{O}(\tilde{r}^4) , & \tilde{r} \to 0, \\ \frac{e^{\zeta'(-1)}}{\sqrt{\pi}\, 2^{91/48}} \exp\left(-\frac{4}{3}\tilde{r}^{3/2} + \frac{8}{3}\sqrt{2}\,\tilde{r}^{3/4}\right) \tilde{r}^{-21/32} \left(1 - \frac{1405\sqrt{2}}{1536\,\tilde{r}^{3/4}} + \mathcal{O}(\tilde{r}^{-3/2})\right) , & \tilde{r} \to +\infty . \end{cases}$$



Red dots : evaluation of $\tilde{p}_{\rm typ}(\tilde{r})$ on a log-log plot, dashed curve : numerical result with 2×10^5 realizations for a 1000×1000 GUE matrix, asymptotic behaviors (black solid line).

4 Conclusion

➪ Exact results for the statistics of near extreme eigenvalues of GUE and asymptotic behaviors.

➪ Another natural quantity to characterize the phenomenon of crowding near the maximum is the (average) density of states [4]:

$$\rho_{\rm DOS}(r,N) = \frac{1}{N-1} \sum_{\substack{i=1 \\ i\neq i_{\max}}}^{N} \langle \delta(\lambda_{\max} - \lambda_i - r) \rangle ,$$

where $i_{\max}$ is such that $\lambda_{i_{\max}} = \lambda_{\max}$ and $\langle \cdots \rangle$ means an average taken with the joint PDF of the eigenvalues. We showed that $\rho_{\rm DOS}(r,N)$ and $p_{\rm GAP}(r,N)$ are actually related via the relation $p_{\rm GAP}(r,N) = (N-1)\rho_{\rm DOS}(-r,N)$.

✍ New formula for the PDF of the first gap in terms of Painlevé transcendents different from the one found previously by Witte, Bornemann and Forrester [3]. It would be interesting to show explicitly that these two formulas do coincide.

✍ Extend to other ensembles of random matrix theory: the Gaussian β-ensembles and hard edge of the Laguerre-Wishart ensemble.

References

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