Moments of β-Ensembles

Alexi Reynolds & Francesco Mezzadri

School of Mathematics, University of Bristol.

Abstract

We present an explicit formula for the moments of the Laguerre β-ensemble. The method of calculation is briefly described and leads to a general formula which we have also successfully applied to the Gaussian and Jacobi β-ensembles (addressed in a forthcoming paper). This is the first time an explicit formula has been found for the moments of β-ensembles.

Introduction

β-ensembles generalise the orthogonal ($\beta = 1$), unitary ($\beta = 2$) and symplectic ($\beta = 4$) random matrix ensembles first introduced by Dyson (referred to as the 'three-fold way'). The entries of a random matrix are weighted by a probability distribution. Random matrices with Gaussian weights have been extensively studied for their applications to physical problems. Also widely studied are the ensembles with Laguerre and Jacobi weights which have applications to multivariate statistics.

The notion of a continuous $\beta > 0$ parameter has been around as long as the classical ensembles themselves. Dyson introduced a log-gas analogy alongside the classical Gaussian ensembles in his seminal paper (1962). This analogy gives an interpretation of the eigenvalues as the particles of a gas subject to a pair-wise repulsion and an external harmonic potential. In the log-gas the parameter β serves as the inverse temperature. A matrix realisation of the β-ensembles was first introduced by Dumitriu and Edelman (2000).

In this work we have made use of the connection that exists between β-ensembles and Jack polynomials. This connection was first discovered in relation to Calogero-Sutherland (C-S) quantum many-body systems. The eigenvalue probability density functions (p.d.f.s) of β-ensembles coincide with the squared ground-state eigenfunctions of certain C-S systems. Furthermore, the higher energy eigenfunctions are given by products of the ground-state eigenfunction with linear combinations of Jack polynomials.

Necessary Symmetric Function Theory

Symmetric functions are invariant under all permutations of their variables. The monomials form a basis for the ring of symmetric functions. They are indexed by an integer partition κ and defined by the sum,

$$m_\kappa\left(\vec{\lambda}\right) = \sum_{\phi \in \mathbf{S}_\kappa} \lambda_{\phi(1)}^{\kappa_1} \lambda_{\phi(2)}^{\kappa_2} \dots \lambda_{\phi(l)}^{\kappa_l}.$$

In this definition S_κ is the set of permutations which gives distinct terms in the sum.

Jack polynomials are orthogonal, multivariate polynomials. They have an explicit formula in terms of the monomials in which the coefficients are determined by recurrence relations. This formula is

$$P_\kappa^{(\alpha)}\left(\vec{\lambda}\right) = m_\kappa\left(\vec{\lambda}\right) + \sum_{\sigma \preceq \kappa} u_{\kappa,\sigma}^{\alpha} m_\sigma\left(\vec{\lambda}\right).$$

We are interested in Jack polynomials with a different normalisation denoted by $C_\kappa^{(\alpha)}\left(\vec{\lambda}\right)$.

In this work we have also made use of a hypergeometric function of matrix argument defined in terms of Jack polynomials. Namely,

$${}_1\mathcal{F}_0^{(\alpha)}\left(b;\vec{x};\vec{y}\right) = \sum_\kappa \frac{[b]_\kappa^{(\alpha)} C_\kappa^{(\alpha)}(\vec{x})\, C_\kappa^{(\alpha)}(\vec{y})}{|\kappa|!\, C_\kappa^{(\alpha)}(1^N)}.$$

The notation $[\dots]_\kappa^{(\alpha)}$ denotes the multivariate Pochhammer symbol.

Overview of Method

We obtain an explicit formula for the moments via a comparison of two series expansions for the integral of the resolvent. One series contains the moments and the other contains the averages of Jack polynomials. The series containing the moments is well-known and the series containing the averages of Jack polynomials can be found via the substitution of a hypergeometric function.

Sketch of Calculation

We want to calculate the moments,

$$\mu_k = \left\langle \sum_{j=1}^N \lambda_j^k \right\rangle.$$

This is not an easy task when the λ_j are the N eigenvalues of a β-ensemble. Instead of attempting a direct calculation we consider the resolvent,

$$W(t) = \left\langle \sum_{j=1}^N \frac{1}{t-\lambda_j} \right\rangle.$$

It is well known that the resolvent is a generating function for the moments. The integral of the resolvent can be written as,

$$\int W(t)\,\mathrm{d}t = N \ln(t) - \sum_{r=1}^{\infty} \frac{\mu_k}{kt^k}. \qquad (1)$$

We can find another series expansion for the integral of the resolvent by rearranging the equation to

$$\int W(t)\,\mathrm{d}t = N\ln(t) + \lim_{n\to 0} \frac{\mathrm{d}}{\mathrm{d}n} \left\langle \prod_{j=1}^N \left(1 - \frac{\lambda_j}{t}\right)^n \right\rangle.$$

The hypergeometric function ${}_1\mathcal{F}_0^{(\alpha)}\left(b;\vec{x};\vec{y}\right)$ satisfies the identity

$${}_1\mathcal{F}_0^{(\alpha)}\left(b; x^N; \vec{\lambda}\right) = \prod_{j=1}^N (1 - x\lambda_j)^{-b}.$$

Using this identity and the definition of the hypergeometric function we can write the integral of the resolvent as

$$\int W(t)\,\mathrm{d}t = N\ln(t) + \sum_\kappa \frac{1}{|\kappa|!}\left(\frac{1}{t}\right)^{|\kappa|} \left(\lim_{n\to 0}\frac{\mathrm{d}}{\mathrm{d}n}[-n]_\kappa^{(\alpha)}\right) \left\langle C_\kappa^{(\alpha)}\left(\vec{\lambda}\right)\right\rangle. \qquad (2)$$

A comparison of (1) and (2) yields the formula for the moments,

$$\boxed{\mu_k = \sum_{\kappa \vdash k} c_\kappa \left\langle C_\kappa^{(\alpha)}\left(\vec{\lambda}\right)\right\rangle}$$

in which

$$c_\kappa = \frac{-1}{(|\kappa|-1)!}\left(\lim_{n\to 0}\frac{\mathrm{d}}{\mathrm{d}n}[-n]_\kappa^{(\alpha)}\right).$$

The limit in c_κ can be evaluated by applying the definition of the multivariate Pochhammer symbol. After some algebraic manipulation we find our final expression for c_κ:

$$c_\kappa = (-1)^{|\kappa|-\kappa_1} \frac{(\kappa_1 - 1)!}{(|\kappa|-1)!} \prod_{i=1}^{l(\kappa)-1} \binom{\frac{i}{\alpha}}{\kappa_{i+1}} \kappa_{i+1}!.$$

Results

Our formula for the moments can be applied to any random matrix ensemble for which the average of a Jack polynomial with respect to the eigenvalue p.d.f. is known. Here we present the results for the Laguerre β-ensemble (LβE) but we have also found analogous results for the Jacobi and Gaussian (Hermite) β-ensembles.

The average of a Jack polynomial with respect to the eigenvalue p.d.f. of the LβE was first calculated by Forrester and Baker whilst studying the relevant C-S system (1997). In the following equations we use the parameter $\alpha = \frac{2}{\beta}$ for ease of notation.

The LβE can be defined by its eigenvalue p.d.f.,

$$P^{(L)}(\vec{\lambda}) = \frac{1}{\mathcal{N}^{(L)}} \prod_{i=1}^N \lambda_i^\gamma e^{-\lambda_i} \prod_{1\le j<k\le N} |\lambda_k - \lambda_j|^{\frac{2}{\alpha}}.$$

The normalisation constant $\mathcal{N}^{(L)}$ can be computed via a Selberg integral and the parameter $\gamma > -1$ is continuous. Using our formula and the results of Forrester and Baker, the moments of the LβE are

$$\mu_k^{(L)} = \sum_{\kappa\vdash k} c_\kappa \left[\gamma + \frac{N-1}{\alpha} + 1\right]_\kappa^{(\alpha)} C_\kappa^{(\alpha)}\left(1^N\right).$$

The moments may also be written in terms of the multivariate Laguerre polynomials $L_\kappa^{\alpha,\gamma}(\vec{\lambda})$ evaluated at $\vec{0}$,

$$\mu_k^{(L)} = \sum_{\kappa\vdash k} c_\kappa L_\kappa^{\alpha,\gamma}(\vec{0}).$$

The multivariate Laguerre polynomials are orthogonal with respect to the eigenvalue p.d.f. of the LβE and they are exactly those polynomials which appear in the eigenfunctions of the C-S system relevant to the LβE.

References

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