Fine Asymptotics in a Random Normal Matrix Ensemble

Roman Riser (joint work with Seung-Yeop Lee, USF)
Department of Mathematics, KU Leuven, Belgium

Definition of the Random Matrix Model

- We consider the probability distribution on $n \times n$ normal matrices
$$\mathcal{P}_n(M) = \frac{1}{Z_n} \mathrm{e}^{-N \operatorname{tr} \mathcal{V}(M)}, \qquad \mathcal{Z}_n = \int_{[M,M^*]=0} e^{-N \operatorname{tr} \mathcal{V}(M)} \mathrm{d}M$$
with the matrix valued potential $\mathcal{V}(M) = M^*M - f(M)^* - f(M)$. The function f has to be such that $\mathcal{Z}_n$ is well-defined. N is a real, positive parameter.
- For the above probability distribution it is well known that the probability distribution on the eigenvalues $\underline{z} = (z_i)_{i=1}^n$ is given by
$$P_n(\underline{z}) = \frac{1}{Z_n} \mathrm{e}^{-N \sum_{i=1}^n V(z_i)} |\Delta(\underline{z})|^2, \quad Z_n = \int_{\mathbb{C}^n} \mathrm{e}^{-N \sum_{i=1}^n V(z_i)} |\Delta(\underline{z})|^2 \mathrm{d}^{2n}\underline{z}$$
where $\Delta(\underline{z}) = \det(z_i^{j-1})_{i,j=1}^n = \prod_{i<j}(z_i - z_j)$ is the Vandermonde determinant and the real-valued potential V is induced by $\mathcal{V}(z\mathbb{1}) = V(z)\mathbb{1}$.
- In the following we will restrict to the case where V is the quadratic potential $V(z) = |z|^2 - t\,\mathrm{Re}(z^2)$ with $0 < t < 1$ and we set $T = \frac{n}{N}$.

Conformal Map and Schwarz Function for the Ellipse

- For quite weak conditions on the potential V it is known that the equilibrium measure has support on a compact domain in the complex plane. For our quadratic potential the eigenvalues fill the inside of the ellipse $\mathcal{E} = \{z \in \mathbb{C} | (\mathrm{Re} z)^2/a_t^2 + (\mathrm{Im} z)^2/b_t^2 = 1\}$ with constant density. The mayor and minor semi-axis are $a_t = \sqrt{\frac{T(1+t)}{1-t}}$ and $b_t = \sqrt{\frac{T(1-t)}{1+t}}$ respectively. Fhe foci at $\pm F_0$ are given by $F_0 = \sqrt{\frac{4tT}{1-t^2}}$.
- The conformal map ϕ from the outside of the unit disc to the outside of the ellipse and its inverse ψ are given by
$$\phi(u) = \frac{F_0}{2\sqrt{t}}\left(u + \frac{t}{u}\right), \qquad \psi(z) = \frac{\sqrt{t}z}{F_0}\left(1 + \sqrt{1 - F_0^2/z^2}\right).$$
The branch-cut of ψ we will call $\mathcal{F} = [-F_0, F_0]$. ψ is a biholomorphic map from $\mathbb{C} \setminus \mathcal{F}$ onto $\mathbb{C} \setminus \overline{B}_{\sqrt{t}}$.
- The Schwarz function S for the ellipse is the unique analytic function on $\mathbb{C} \setminus \mathcal{F}$ with $S(z) = \bar{z}$ on $\mathcal{E}$. It is given by
$$S(z) = t\,z + \frac{2T\,z}{F_0^2}\left(1 - \sqrt{1 - F_0^2/z^2}\right), \qquad z \in \mathbb{C} \setminus \mathcal{F}.$$

Orthonormal Polynomials

- For $k \in \mathbb{N}_0$ we define the orthonormal polynomials p_k of degree k regarding the inner product
$$(p_k, p_l) = \int_{\mathbb{C}} \overline{p_k(z)} p_l(z) e^{-NV(z)} \mathrm{d}^2 z.$$
- Let $r \in \mathbb{Z}$ and $z \in \mathbb{C} \setminus \mathcal{F}$. As $n, N \to \infty$ the orthonormal polynomial have the asymptotics
$$p_{n+r}(z) = \left(\frac{N}{2\pi^3}\right)^{1/4} \mathrm{e}^{-\frac{N\ell}{2}} \mathrm{e}^{ng(z)} \sqrt{\psi'(z)} \psi(z)^r \left(1 + \frac{h_{n+r}}{N} + \mathcal{O}(N^{-2})\right),$$
where the error term of order $\mathcal{O}(N^{-2})$ holds uniformly on $\mathbb{C} \setminus \mathcal{F}$. The g-function is given by
$$g(z) = \frac{\left(1 - \sqrt{1 - F_0^2/z^2}\right) z^2}{F_0^2} + \log\left(z + z\sqrt{1 - F_0^2/z^2}\right) - \frac{1}{2} - \log 2.$$
The correction h_{n+r} of order $\mathcal{O}(N^{-1})$ to the polynomial (we only need h_n and h_{n-1}) are given by
$$h_n(z) = \frac{3F_0^2\left(2\sqrt{1-F_0^2/z^2} - 1\right) - 2z^2}{48T\sqrt{1-F_0^2/z^2}(z^2 - F_0^2)}, \qquad h_{n-1}(z) = \frac{3F_0^2\left(-2\sqrt{1-F_0^2/z^2} - 1\right) - 2z^2}{48T\sqrt{1-F_0^2/z^2}(z^2 - F_0^2)}.$$
Robin's constant is given by $\ell = T\left(\log\left(\frac{F_0^2}{4t}\right) - 1\right)$.

Normalized Reproducing Kernel

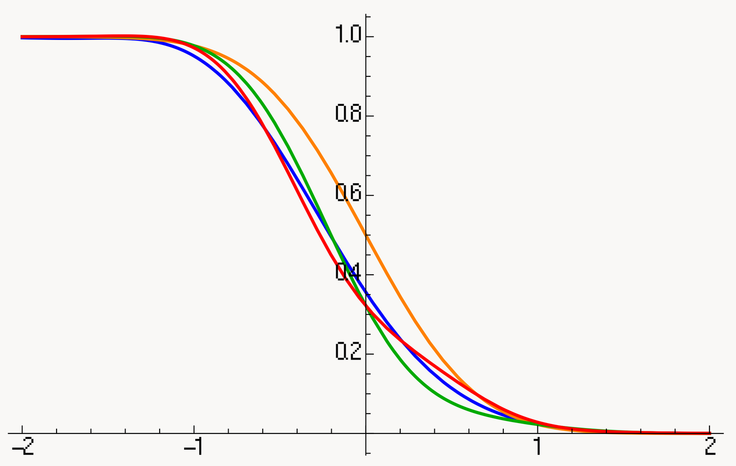
- With the help of the orthonormal polynomials we can define the reproducing kernel
$$K_n(w, z) = \mathrm{e}^{-\frac{N}{2}(V(w)+V(z))} \sum_{k=0}^{n-1} \overline{p_k(w)} p_k(z).$$
- The average density of eigenvalues is given by $\rho_n(z) = \frac{1}{n} K_n(z, z)$. The density is normalized to 1.

Main Result

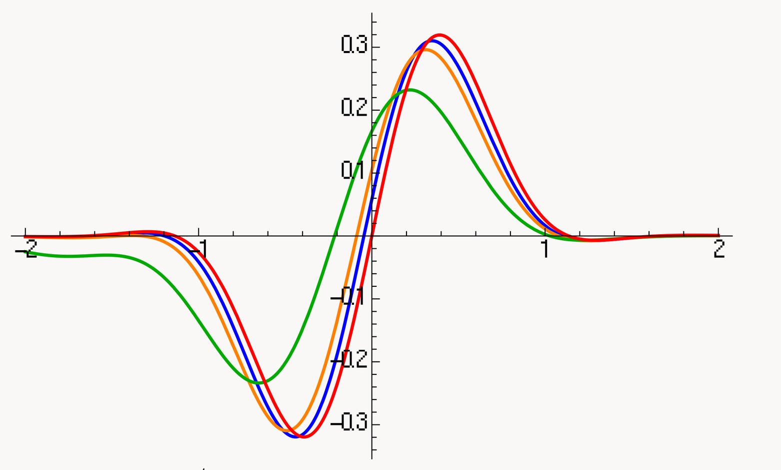
- **Theorem (S. Y. Lee, R. Riser)** For $z \in \mathbb{C} \setminus \mathcal{E}$ the density of eigenvalues has only exponentially small correction terms, while on $\mathcal{E}$ the first correction is of order $\mathcal{O}(N^{-1/2})$. This means there exists $\epsilon > 0$ small enough so that when $N, n \to \infty$
$$\rho_n(z) = \begin{cases} \frac{1}{\pi T} + \mathcal{O}(\mathrm{e}^{-N\epsilon}) & \text{for } z \in \mathrm{Int}(\mathcal{E}), \\ \frac{1}{2\pi T} - \frac{\kappa}{3T\sqrt{2\pi^3 N}} + \mathcal{O}(N^{-3/2}) & \text{for } z \in \mathcal{E}, \\ \mathcal{O}(\mathrm{e}^{-N\epsilon}) & \text{for } z \in \mathrm{Ext}(\mathcal{E}). \end{cases}$$
- **Theorem (S. Y. Lee, R. Riser)** Let $z_0 \in \mathcal{E}$, $A > 0$, $\alpha, \beta \in \mathbb{R}$ with $|\alpha| + |\beta| < AN^{1/6}$. The outer normal vector to the ellipse $\mathcal{E}$ at z_0 we denote by $\mathbf{n} = n_x + in_y \in \mathbb{C}$ with $|\mathbf{n}| = 1$ and the curvature of $\mathcal{E}$ at z_0 by $\kappa > 0$. The first and second derivative of κ regarding arclength we denote by $\partial_s\kappa$ and $\partial_s^2\kappa$. Then for large N the average density of eigenvalues at $z_0 + \frac{(\alpha+i\beta)\mathbf{n}}{\sqrt{N}}$ is given by
$$\begin{aligned}\rho_n\left(z_0 + \frac{(\alpha+i\beta)\mathbf{n}}{\sqrt{N}}\right) = \frac{\operatorname{erfc}(\sqrt{2}\alpha)}{2\pi T} + \frac{\mathrm{e}^{-2\alpha^2}}{T\sqrt{2\pi^3}} \Bigg[& \frac{\kappa}{\sqrt{N}}\left(\frac{\alpha^2 - 1}{3} - \beta^2\right) \\ & + \frac{\kappa^2}{N} \frac{2\alpha^5 - 8\alpha^3 + 3\alpha + 36\alpha\beta^2 - 12\alpha^3\beta^2 + 18\alpha\beta^4}{18} \\ & + \frac{1}{N}\left(\left(\frac{(\partial_s\kappa)^2}{9\kappa^2} - \frac{\partial_s^2\kappa}{12\kappa}\right)\alpha + \frac{\partial_s\kappa}{3}\left(\alpha^2\beta - \beta^3 - \beta\right)\right) + \mathcal{O}\left((|\alpha| + |\beta|)^8 N^{-3/2}\right) \Bigg]. \end{aligned} \tag{1}$$
The error term $\mathcal{O}\left((|\alpha| + |\beta|)^8 N^{-3/2}\right)$ holds uniformly for $|\alpha| + |\beta| < AN^{1/6}$.

Plot of Density at the Boundary

We denote by ρ^k the density as in (1) up to order $\mathcal{O}(N^{-k})$. The following plots compare numerical computations of the density ρ_n for finite n with our result. The density is plotted at the boundary in local coordinate in normal direction. We have chosen $T = 1$, $t = 1/2$, $z_0 = \sqrt{3}$, $\kappa(z_0) \approx 5.2$. The left-hand side shows the density multiplied by π for $n = 20$. On the right-hand side we have subtracted $\rho^{1/2}$ from the density and multiplied by N.



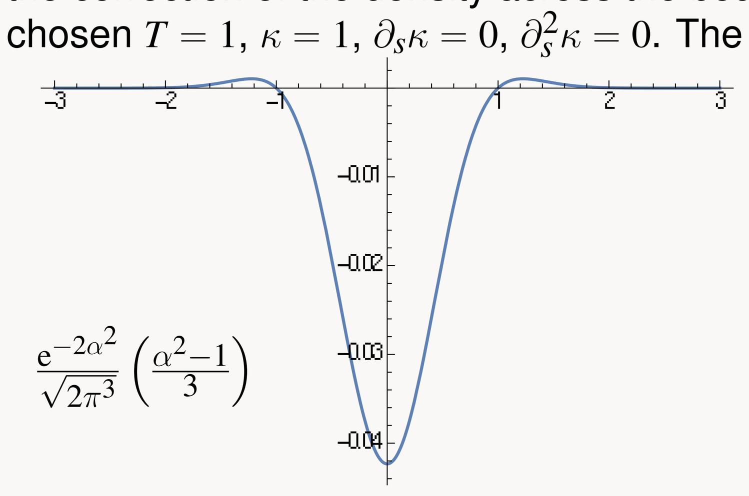
$\pi\rho_n(z)$ (blue), $\pi\rho^0$ (orange), $\pi\rho^{1/2}$ (green), $\pi\rho^1$ (red).

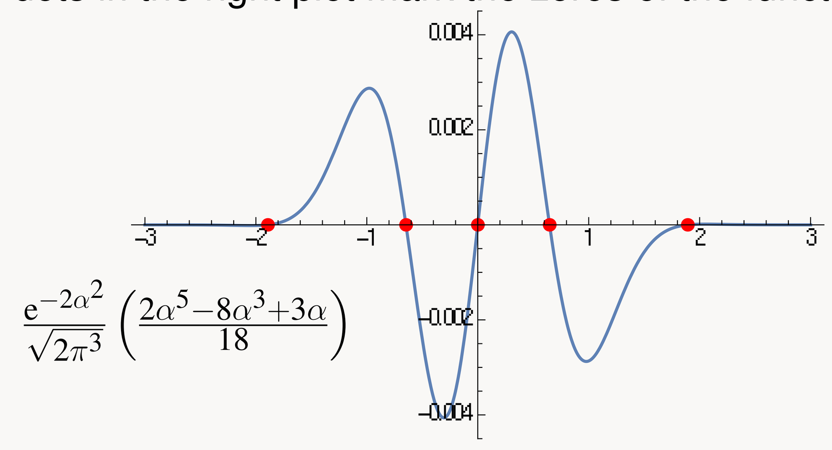


$N(\rho - \rho^{1/2})$: $\rho = \rho_n$ for $N = 10$ (green), $N = 100$ (orange), $N = 400$ (blue), $\rho = \rho^1$ (red).

Plot of Correction to the Density

These plots show $\sqrt{N}(\rho^{1/2} - \rho^0)$ (left-hand side) and $N(\rho^1 - \rho^{1/2})$ (right-hand side). The plots show the correction of the density across the boundary, in local coordinates in normal direction. We have chosen $T = 1$, $\kappa = 1$, $\partial_s\kappa = 0$, $\partial_s^2\kappa = 0$. The red dots in the right plot mark the zeros of the function.





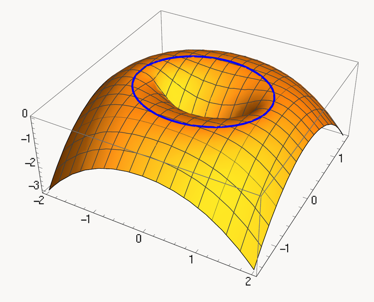
Identities for κ in terms of S

We have the following identities for the normal vector, the curvature and its derivative in terms of the Schwarz function. They not only hold for the ellipse but for any Schwarz function associated to a curve. Let z_0 be a point on the curve. Then

- $\mathbf{n}^2 S'(z_0) = -1$,
- $\kappa = -\mathbf{n}\frac{S''(z_0)}{2S'(z_0)}$,
- $\partial_s\kappa = \frac{i}{4}\left(\frac{2S'''(z_0)}{(S'(z_0))^2} - \frac{3(S''(z_0))^2}{(S'(z_0))^3}\right)$,
- $\partial_s^2\kappa = -\frac{\mathbf{n}}{4}\left(\frac{2S^{(4)}(z_0)}{(S'(z_0))^2} - \frac{10S''(z_0)S'''(z_0)}{(S'(z_0))^3} + \frac{9(S''(z_0))^3}{(S'(z_0))^4}\right)$.

Sketch of our Proof

- Our approach is based on the following Christoffel-Darboux like formula.
$$\frac{1}{n}\frac{\partial K_n}{\partial z}(z, z) = \left(-p_n(\bar{z})p_{n-1}(z) + t\,p_{n-1}(\bar{z})p_n(z)\right)\frac{\mathrm{e}^{-NV(z)}}{\sqrt{T(1-t^2)}}.$$
- Putting the asymptotics for the orthonormal polynomials in it we get the following asymptotics of the derivative of the density in the direction $\mathbf{v}$.
$$\partial_{\mathbf{v}}\rho_n(z) = \sqrt{\frac{N}{2\pi^3}}\mathrm{e}^{-N\Omega(z)} 2\mathrm{Re}\left(\frac{\bar{\mathbf{v}}t - \mathbf{v}}{\sqrt{T(1-t^2)}}\frac{|\psi'(z)|}{\psi(z)}\left(1 + \frac{\overline{h_n(z)} + h_{n-1}(z)}{N} + \mathcal{O}(N^{-2})\right)\right)$$
where $\Omega(z) = V(z) - 2T\mathrm{Re}g(z) + \ell$. Ω has a strict global minimum on $\mathcal{E}$ where it takes the value 0. From this it is obvious that away from the boundary the density must be flat up to an error term of order $\mathcal{O}(\mathrm{e}^{-N\epsilon})$ for $\epsilon > 0$ small enough.



Plot of $-\Omega$ for $T = 1$ and $t = -2 + \sqrt{5}$.

- At the boundary $z_0 \in \mathcal{E}$ the expansion for Ω in normal direction $\mathbf{n}$ and tangential direction $\mathbf{t} = i\mathbf{n}$ when X, Y is of order $\mathcal{O}(N^{-1/2})$ is given by
$$\Omega(z_0 + X\mathbf{n} + Y\mathbf{t}) = 2X^2 - \tfrac{2\kappa}{3}(X^3 - 3XY^2) + \tfrac{\kappa^2}{2}(X^4 - 6X^2Y^2 + Y^4) - \tfrac{2\partial_s\kappa}{3}(X^3Y - XY^3) + \mathcal{O}(N^{-5/2}).$$
We also need an analogous expansion for $|\psi'(z)|/\psi(z)$.
- Then we can find the density by the following integration stating at a point z_1 inside of the ellipse.
$$\rho_n\left(z_0 + \frac{\alpha\mathbf{n} + \beta\mathbf{t}}{\sqrt{N}}\right) = \rho_n(z_1) + \int_{z_1}^{z_0 + \frac{\alpha\mathbf{n}}{\sqrt{N}}} \partial_{\mathbf{n}}\rho_n(\xi)\mathrm{d}\xi + \int_{z_0 + \frac{\alpha\mathbf{n}}{\sqrt{N}}}^{z_0 + \frac{\alpha\mathbf{n} + \beta\mathbf{t}}{\sqrt{N}}} \partial_{\mathbf{t}}\rho_n(\xi)\mathrm{d}\xi.$$
- We already know form the former argument that the density inside and outside of the ellipse is flat. So the density has a jump across the boundary of $\frac{1}{T\pi} + \mathcal{O}(\mathrm{e}^{-N\epsilon})$. But when we do the same calculation by integration of $\partial_{\mathbf{n}}\rho_n$ we will get a term of order $\mathcal{O}(N^{-1}$ which contains h_n and h_{n-1}. This term has to be zero which this gives us the identity
$$\mathrm{Re}\left(h_n(z_0) + h_{n-1}(z_0)\right) + \frac{\partial_s\kappa}{3|\psi'(z_0)|\kappa}\mathrm{Im}\left(h_n(z_0) - h_{n-1}(z_0)\right) + \frac{\kappa^2}{12} + \frac{(\partial_s\kappa)^2}{18\kappa^2} - \frac{\partial_s^2\kappa}{24\kappa} = 0. \tag{2}$$

Geometrical Argument

- **Proposition** Let $\epsilon > 0$ small, $a, b \in \mathbb{R}$ of order $\mathcal{O}(\epsilon)$, Γ an analytic curve without self-intersections, which is parametrized as $z(s)$ by arclength s. $z_0 = z(0)$ is a point on Γ.
Then there exists a point $\hat{z}_0 \in \Gamma$ and $\hat{a}$ of order $\mathcal{O}(\epsilon)$ so that $z_0 + a\mathbf{n}(z_0) + ib\mathbf{n}(z_0) = \hat{z}_0 + \hat{a}\mathbf{n}(\hat{z}_0)$ where $\mathbf{n}(z_0)$ and $\mathbf{n}(\hat{z}_0)$ are the oriented unit normal vector at z_0 and $\hat{z}_0$ respectively. Further $\hat{a}$ is given by
$$\hat{a} = a + \frac{\kappa(z_0)b^2}{2} + \frac{\partial_s\kappa(z_0)b^3 - 3(\kappa(z_0))^2ab^2}{6} + \mathcal{O}(\epsilon^4)$$
where $\kappa(z_0)$ is the curvature of Γ at z_0. $\hat{z}_0$ is given by $\hat{z}_0 = z(\hat{s})$ where $\hat{s} = b - \kappa(z_0)ab + \mathcal{O}(\epsilon^3)$. The curvature at $\hat{z}_0$ is related to the curvature at z_0 by
$$\kappa(\hat{z}_0) = \kappa(z_0) + \partial_s\kappa(z_0)\left(b - \kappa(z_0)ab\right) + \frac{b^2\partial_s^2\kappa}{2} + \mathcal{O}(\epsilon^3).$$
- Using this geometrical identity to express the density at the same point in two different ways, we can calculate the tangential contribution from a law for the normal direction. By comparing the two different calculation for the tangential contribution we get the identity
$$\mathrm{Im}\left(h_n(z_0) - h_{n-1}(z_0)\right) = \frac{|\psi'(z_0)|\partial_s\kappa}{6\kappa}.$$
When we use this relation in (2) we can additionally get the identity
$$\mathrm{Re}\left(h_n(z_0) + h_{n-1}(z_0)\right) = -\frac{1}{12}\kappa^2 + \frac{1}{24}\frac{\partial_s^2\kappa}{\kappa}.$$
While these identities only give the real part of the sum or the imaginary part of the difference of h_n and h_{n-1} and only hold on $\mathcal{E}$, we can get $h_n \pm h_{n-1}$ on all of $\mathbb{C} \setminus F$ by Wiener-Hopf decomposition. This is an alternative way to calculate h_n and h_{n-1}..

Total Mass Contribution and Numbers of Eigenvalues

- At the boundary in local coordinates the leading term of the density is given by the erfc. Therefore some eigenvalues get moved from inside of the ellipse to outside. Because of the curvature this gives the following additional mass per unit arclength
$$\frac{1}{2\pi T}\int_{-\infty}^{\infty}\left(\operatorname{erfc}(\sqrt{2}\alpha) - 2\theta(-\alpha)\right)\left(\frac{1}{\kappa} + \frac{\alpha}{\sqrt{N}}\right)\mathrm{d}\alpha = \frac{1}{8\pi T\sqrt{N}}.$$
This will be balanced by the subleading correction term
$$\int_{-\infty}^{\infty}\frac{\mathrm{e}^{-2\alpha^2}\kappa}{3T\sqrt{2\pi^3}N}(\alpha^2 - 1)\left(\frac{1}{\kappa} + \frac{\alpha}{\sqrt{N}}\right)\mathrm{d}\alpha = -\frac{1}{8\pi T\sqrt{N}}.$$
- We can also calculate the numbers of eigenvalues outside of the ellipse per unit arclength on the boundary which is given by
$$n\int_0^{\infty}\rho_n(z_0 + X\mathbf{n})\left(\frac{1}{\kappa} + \frac{\alpha}{\sqrt{N}}\right)\mathrm{d}X = \frac{\sqrt{N}}{2\sqrt{2\pi^3}\kappa}\left(1 + \frac{1}{6N}\left(-\frac{\kappa^2}{2} + \frac{(\partial_s\kappa)^2}{\kappa^2} - \frac{\partial_s^2\kappa}{4\kappa}\right) + \mathcal{O}(N^{-3/2})\right).$$

References

1. P. Elbau and G. Felder. Density of eigenvalues of random normal matrices. *Communications in Mathematical Physics*, 259:433–450, 2005.
2. R. Riser. Universality in Gaussian Random Normal Matrices. *ArXiv e-prints*, 1312.0068, 2013.
3. P. Bleher and A. Kuijlaars. Orthogonal polynomials in the normal matrix model with a cubic potential. *Advances in Mathematics* 230.3: 1272-1321, 2012.
4. F. Balogh, M. Bertola, S. Y. Lee, and K. McLaughlin. Strong asymptotics of the orthogonal polynomial with respect to a measure supported on the plane. *ArXiv e-prints*, 1209.6366, 2012.
5. Y. Ameur, N. Kang, and N. Makarov. Rescaling Ward identities in the random normal matrix model. *ArXiv e-prints*, 1410.4132, 2014.

roman.riser@wis.kuleuven.be