Products with truncated unitary matrices

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Introduction

- Recently, it became clear that in some cases the squared singular values of a product of random matrices still had a determinantal structure. For example, in the case of a product of Ginibre matrices, it was shown in [4, 3] that the squared singular values are a determinantal point process with joint p.d.f. proportional to
$$\prod_{j<k}(y_k - y_j)\det\left[g_{k-1}(y_j)\right]_{j,k=1}^n$$
where $g_k(y)$ is a Meijer G-function
- This was put in a more general framework showing that if a random matrix X had a particular determinantal structure the product matrix GX, with G a Ginibre random matrix, had the same structure [5]
- Two key ingredients for this proof:
  - Explicit formula for the distribution of G, namely $e^{-\mathrm{Tr}(G^*G)}dG$
  - Harish-Chandra/Itzykson-Zuber integral formula
- **Question**: could we replace G by another random matrix such that the structure would still be preserved?

The random matrix model

- Random matrix X of size $l \times n$, with $l \geq n$
- Squared singular values of X have j.p.d.f.
$$\propto \prod_{j<k}(x_k - x_j)\det\left[f_k(x_j)\right]_{j,k=1}^n$$
- U a Haar distributed random unitary matrix of size $m \times m$
- T the $(n+\nu)\times l$ upper left submatrix of U

Main result

Let X and T be as above. Then the squared singular values of $Y := TX$ have j.p.d.f.
$$\propto \prod_{j<k}(y_k - y_j)\det\left[g_k(y_j)\right]_{j,k=1}^n$$
where
$$g_k(y) = \int_0^1 x^{\nu}(1-x)^{m-n-\nu-1} f_k\left(\frac{y}{x}\right)\frac{dx}{x}$$
which is the Mellin convolution of f_k with a beta distribution.

Proof: First approach

For this approach we have to assume $m \geq 2n+\nu$. In this case there is an explicit formula for the distribution of a truncation of size $(n+\nu)\times n$ which we can use.

0. We may restrict to the case $l = n$.

Keep X fixed:

1. Make the change of variables $T \mapsto Y = TX$
2. Make the change of variables to the singular value decomposition $Y = U\Sigma V$
3. Integrate U and V over the unitary group. HCIZ-analogue integral formula:

Integral over unitary group

Let A and B be $n\times n$ Hermitian matrices with respective eigenvalues $a_1,\dots,a_n$ and $b_1,\dots,b_n$. Let dU be the normalized Haar measure on the unitary group $\mathcal{U}(n)$. Then for every $p \geq 0$,
$$\int_{\mathcal{U}(n)} \det\left(A - UBU^*\right)^p \theta(A-UBU^*)\,dU = c_{n,p}\frac{\det\left[(a_j-b_k)_+^{p+n-1}\right]_{j,k=1}^n}{\Delta(a)\Delta(b)}$$

4. Look at the joint p.d.f. of the squared singular values of Y given those of X

Let X vary:

5. Integrate out the squared singular values of X

Proof: Second approach

Our aim: compute $\mathbb{E}\left[f(YY^*)\right]$ for every continuous symmetric function f on $[0,\infty)^n$.

Key ingredients:

0. $F(M) := f(M_1XX^*M_1)$ with M_1 the $(n+\nu)\times n$ upper left block of M

$\Rightarrow$ $\mathbb{E}\left[f(YY^*)\right] = \int_{\mathcal{U}(n)} F(U)dU$

1. Lift integration over unitary group to integration over all matrices
   - $dU = p(M)dM$
   - $p(M) = c\int_{H(m)} e^{i\mathrm{Tr}H(MM^*-I)}dH$

$\Rightarrow$ $\int_{\mathcal{U}(n)} F(U)dU = c\int_{H(m)} dH \int_{M(m)} dM F(M) e^{i\mathrm{Tr}H(MM^*-I)}$

2. Gaussian matrix integrals: If A is an $n\times n$ matrix with positive-definite Hermitian part and J an $m\times n$ matrix, then
$$\int_{M(m,n)} e^{-\mathrm{Tr}(AM^*M - iJ^*M - iM^*J)}dM = \frac{(2\pi)^{mn}}{(\det A)^n}e^{-\mathrm{Tr}(JA^{-1}J^*)}$$
3. A shift in the integral over Hermitian matrices: For a particular function Φ and for any matrix A with $\mathrm{Im}\ A > -I$ we have
$$\int_{H(n)}\Phi(K+A)dK = \int_{H(n)}\Phi(K)dK$$

Product of r truncated unitary matrices

$$Y_r = T_r \dots T_1$$

- T_j is the $(n+\nu_j)\times(n+\nu_{j-1})$ truncation of an $m_j\times m_j$ Haar distributed unitary matrix where $m_1 \geq 2n+\nu_1$
- Repeated use of our main result tells us that the squared singular values of Y_r have a joint p.d.f.
$$\propto \prod_{j<k}(y_k-y_j)\det\left[w_k^{(r)}(y_j)\right]$$
where the weight functions are obtained as an $(r-1)$-fold Mellin convolution of beta distributions
- $w_k^{(r)}(y) = G^{r,0}_{r,r}\left(\begin{matrix} m_r - n, \dots, m_2 - n, m_1 - 2n + k + 1 \\ \nu_r, \dots, \nu_2, \nu_1 + k\end{matrix}\,\middle|\, y\right)$

$\Rightarrow$ $K_n(x,y) = \dfrac{1}{(2\pi i)^2}\displaystyle\int_{-1/2-i\infty}^{-1/2+i\infty} ds \oint_\Sigma dt \prod_{i=0}^r \frac{\Gamma(s+1+\nu_i)}{\Gamma(t+1+\nu_i)}\frac{\Gamma(t+1+m_i-n)}{\Gamma(s+1+m_i-n)}\frac{x^t y^{-s-1}}{s-t}$

Scaling limit at the origin (hard edge)

$$\lim_{n\to\infty}\frac{1}{\prod_{i=1}^r(m_i-n)n}K_n\left(\frac{x}{\prod_{i=1}^r(m_i-n)n}, \frac{y}{\prod_{i=1}^r(m_i-n)n}\right) = K_{\nu_1,\dots,\nu_r}(x,y)$$

- $K_{\nu_1,\dots,\nu_r}(x,y) = \dfrac{1}{(2\pi i)^2}\displaystyle\int_{-1/2-i\infty}^{-1/2+i\infty} ds \oint_\Sigma dt \prod_{i=0}^r \frac{\Gamma(s+1+\nu_i)}{\Gamma(t+1+\nu_i)}\frac{\sin(\pi s)}{\sin(\pi t)}\frac{x^t y^{-s-1}}{s-t}$
- $r = 1$: Bessel kernel

Future research

Do there exist analogous "transformation formulas" concerning the distribution of the **eigenvalues** of the product of random matrices?

- Product of r Ginibre random matrices was studied in [1]
- Product of r truncated unitary matrices was studied in [2]

[1] G. Akemann and Z. Burda, Universal microscopic correlation functions for products of independent Ginibre matrices, *J. Phys. A* 45 (2012), 465201, 18 pp.

[2] G. Akemann, Z. Burda, M. Kieburg, and T. Nagao, Universal microscopic correlation functions for products of truncated unitary matrices, preprint arXiv: 1310.6395.

[3] G. Akemann, J.R. Ipsen, and M. Kieburg, Products of rectangular random matrices: singular values and progressive scattering, *Phys. Rev. E* 88 (2013), 052118, 13 pp.

[4] G. Akemann, M. Kieburg, and L. Wei, Singular value correlation functions for products of Wishart random matrices *J. Phys. A* 46 (2013), 275205, 22 pp.

[5] A. Kuijlaars and D. Stivigny, Singular values of products of random matrices, Random Matrices: Theory and Applications 3 (2014), no. 3, 1450011

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