

Abstract

Our goal is description of local eigenvalue statistics of invariant random matrix models. We consider $N \times N$ random matrices, governed by a confining analytic potential $V(H)$, and study the characteristic function in the $N \rightarrow \infty$ limit. We use supersymmetry method to obtain behavior of many-point correlation functions. An important lemma, existence and uniqueness of supersymmetric Laplace Transform, has been proven for the functions in question. This technique is a powerful method of determining universality classes in correlations of eigenvalues.

Introduction

Invariant random matrix ensembles:

$$d\mu_N(H) \propto e^{-N\text{Tr}V(H)} dH. \quad (1)$$

Characteristic function:

$$\Omega(K) = \int e^{\text{Tr}KH} d\mu(H). \quad (2)$$

For analytic and convex V we have saddle point equation [1]:

$$Q^{-1} + R(Q) = z\text{Id}_{p|q} \quad (3)$$

where Q is a rank $(p|q)$ supermatrix argument of lifted $\hat{\Omega}(Q)$ required by supersymmetry method and R is a free probabilistic R-transform. Correlation functions may be retrieved from $\hat{\Omega}$ by analog of Laplace Transform:

$$\int \frac{\prod_{b=1}^q \text{Det}(w_{1,b} - H)}{\prod_{a=1}^p \text{Det}(w_{0,a} - H)} d\mu_N(H) \propto \int \text{SDet}^N(Q) \hat{\Omega}(Q) e^{-\text{STr}wQ} DQ \quad (4)$$

Objectives

1. Establish existence and uniqueness of Laplace Transform
2. Analyze singularities of R-transform
3. Obtain minimal requirements for the formalism
4. Determine universality classes in correlations
5. Compute interesting models
6. Extend to random matrices with non-real spectrum

Supersymmetric Laplace Transform

$$\tilde{F}(P) = \frac{1}{(2\pi i)^{n^2+m^2}} \int_{U(n) \times iH^+(m)} e^{-\text{STr}PQ} F(Q) DQ. \quad (5)$$

Starting point: Fourier Transform - additional factor "i" in the exponent, integration done over $H(n) \times H(m)$.

References

- [1] S. Mandt and M.R. Zirnbauer, J. Phys. A 43 025201, (2010)
[2] B. Eynard, JHEP 0411 031, (2004)

Fermion-Fermion sector

$$F(Q) = \prod_j \text{Det}(q_j - H)$$

1. Splitting integration over $H(n)$ to sum of integrals over all possible signatures.
2. Analytic continuation to match each integration region with $H_+(n)$:

$$\hat{F}(P) = \lim_{\eta \rightarrow 0} \sum_{S=\text{diag}(\pm 1)} (-1)^{\text{Tr}S} \tilde{F}(i\Lambda_P + \eta S). \quad (6)$$

3. Inverse by change of variables and closing contours of integration in $\pm i\infty$ (Fig. (1)).

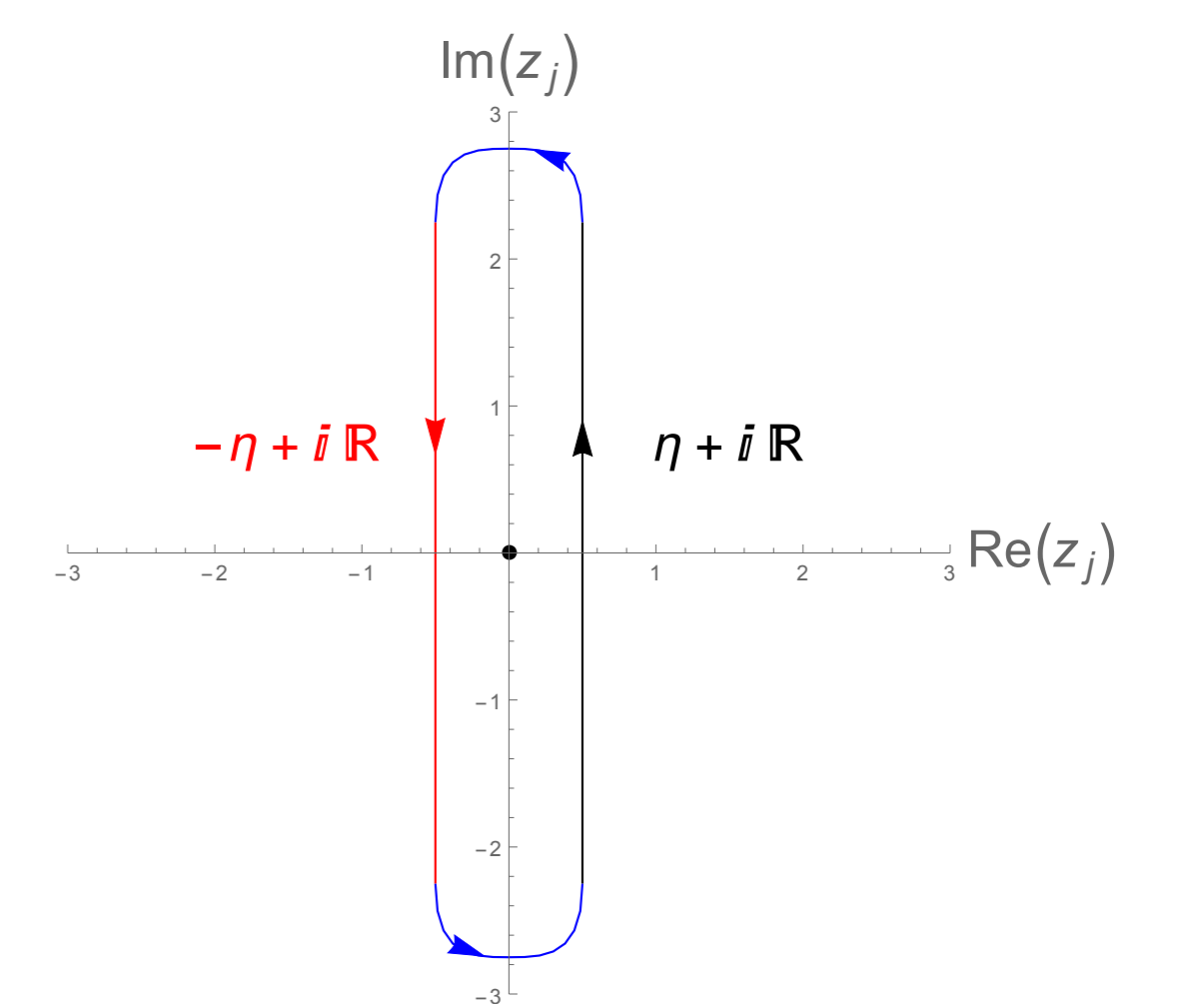


Figure 1: Integration contour for one of the eigenvalues in Fermion-Fermion sector for Inverse Laplace Transform

Boson-Boson sector

$$F(Q) = \prod_j \text{Det}^{-1}(q_j - H)$$

1. Shifting poles from \mathbb{R} to \mathbb{C} by $\pm i\epsilon$.
2. Closing contour of integration in upper/lower complex plane (Fig. (2)).
3. Deforming the contour to match $U(m)$ integration region.
4. Inverse by simple change of variables: $P' = iP$

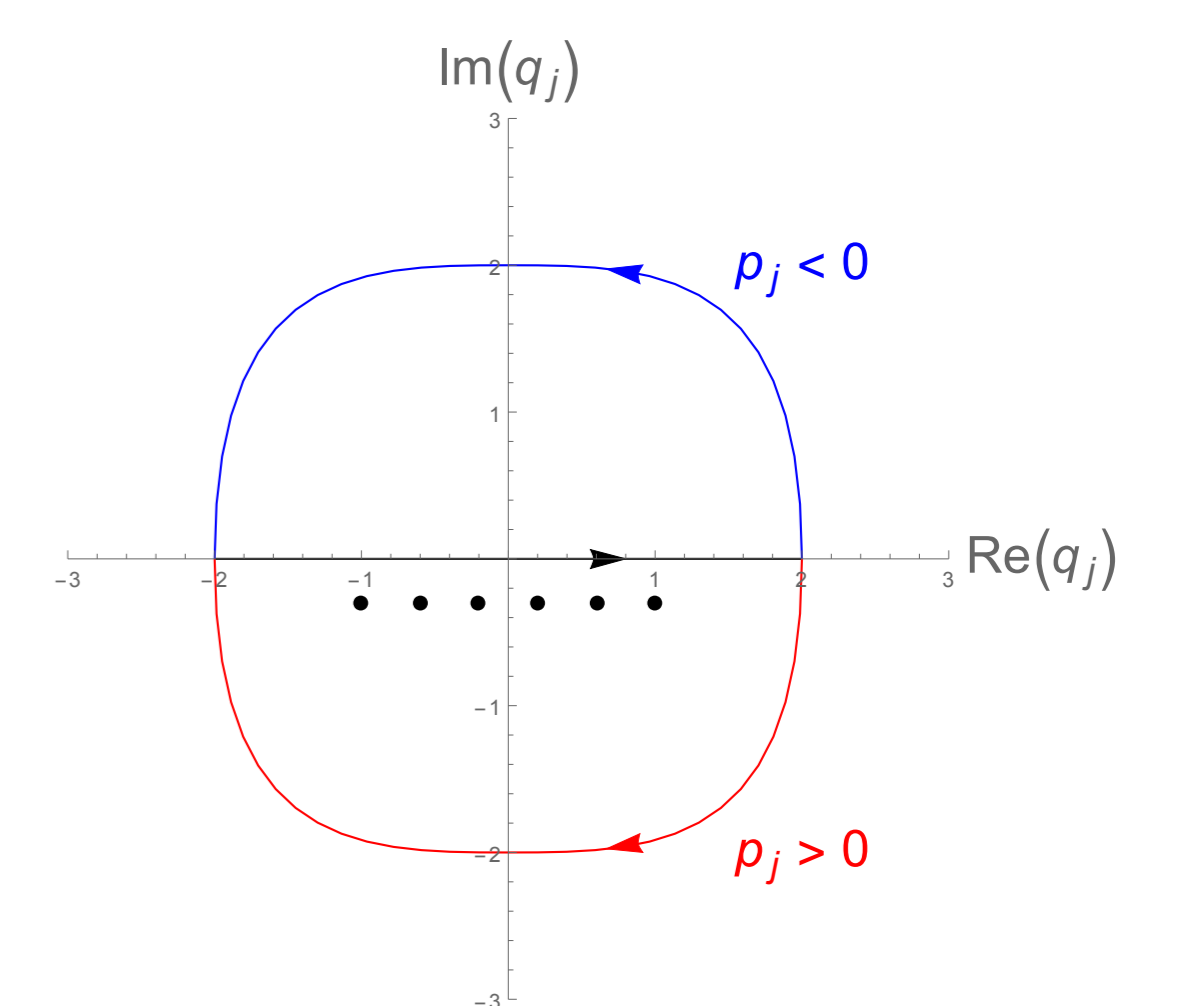


Figure 2: Integration contour for one of the eigenvalues in Boson-Boson sector for Laplace Transform

Singularities of R-transform

For uniformly convex potentials V , the R-transform is analytic in whole complex plane. Once the convexity is lost, singularities may appear. Information about singularities is crucial when considering correlation functions. One may restrict the location of the singular values via Implicit Function Theorem. Instead of looking at singularities of $R(g(z))$ we search for zeroes of $g'(z)$.

Region

If the eigenvalue density is supported on interval $[a, b]$, singularities are restricted to the circle with diameter $a-b$ and center at $(a+b)/2$. Each hole in the density in the similar manner excludes a circle from the region.

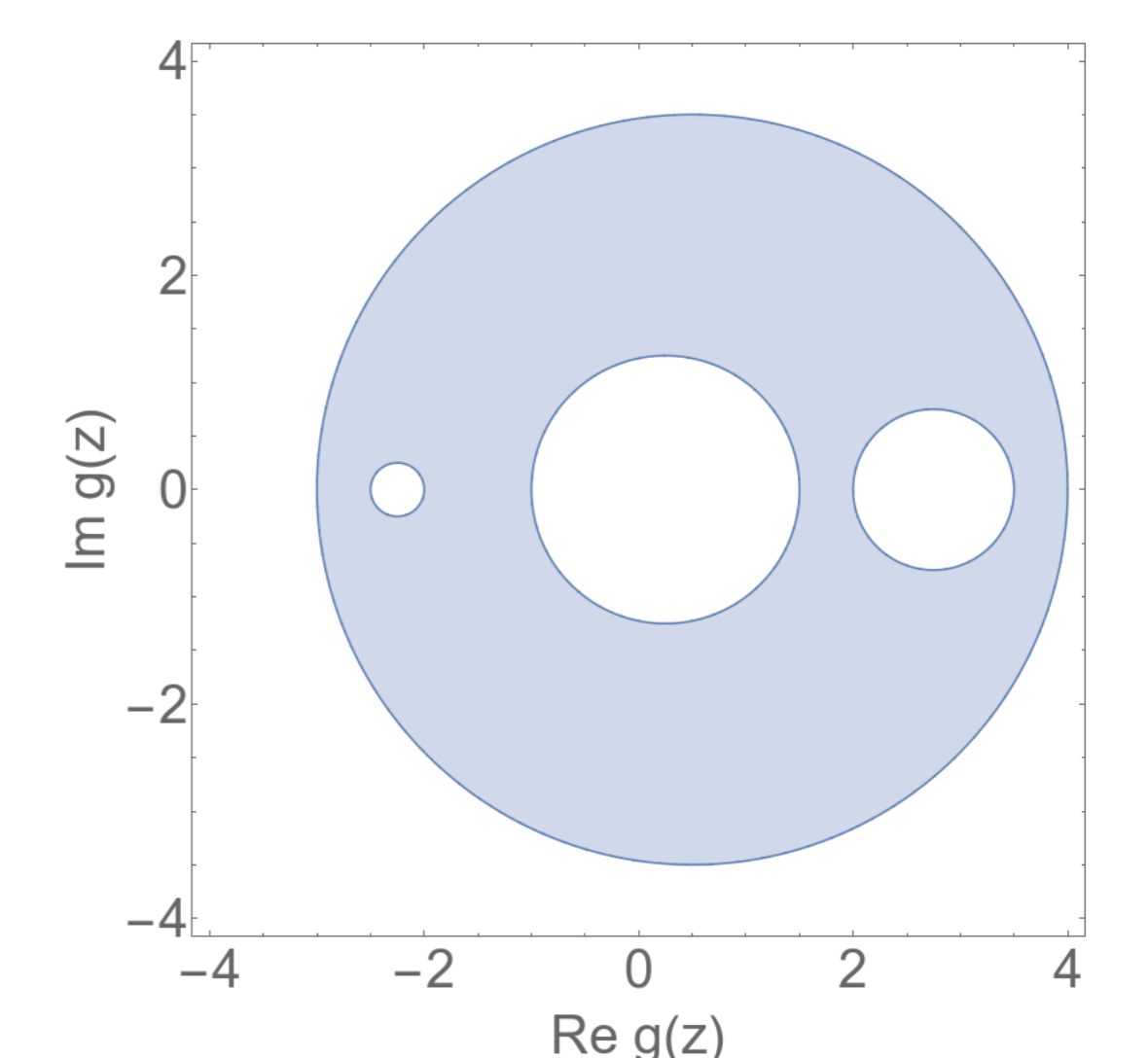


Figure 3: Allowed region for singularities of R-transform when eigenvalue density is supported on $[-3.0, -2.5] \cup [-2.0, -1.0] \cup [1.5, 2.0] \cup [3.5, 4.0]$

Birth of singular values

When considering continuous deformation of convex potential to non-convex one, the singularities appear in even numbers in the neighborhood of real line ($z \rightarrow \bar{z}$ symmetry). For analytic potentials, the condition for a pair of singular values to be born near $x \in \mathbb{R}$ is:

$$V''(x) = \rho'(x) = 0. \quad (7)$$

In special case of $V(z) = V(-z)$, singularities appear when convexity of V changes near $z = 0$.