Joint distribution of two linear statistics in the Laguerre ensemble. Complex impedance of chaotic cavities

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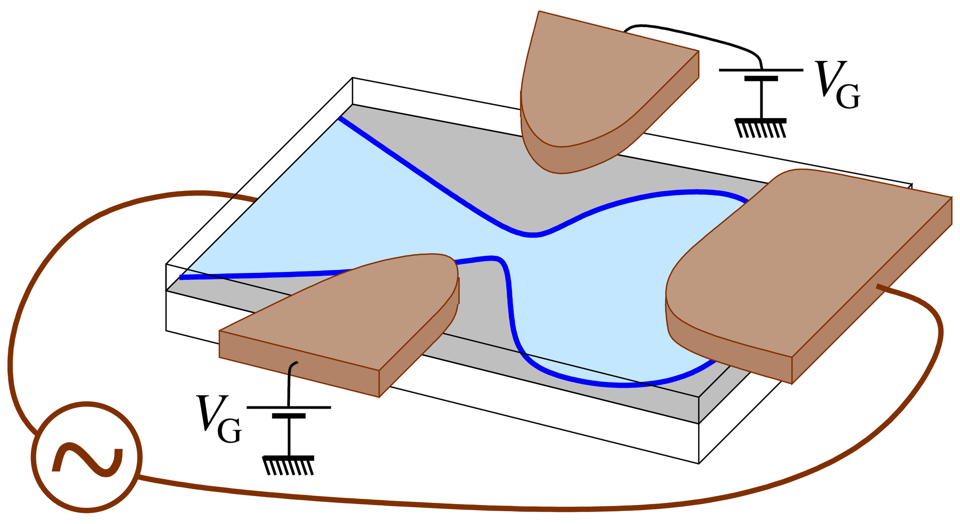
The problem



Coherent AC transport :

$$Z(\omega) = \frac{1}{-\mathrm{i}\omega\, C_\mu} + R_q + \mathcal{O}(\omega) \qquad (1)$$

Büttiker, Prêtre & Thomas (1993) :

$$C_\mu^{-1} = C^{-1} + C_q^{-1} \text{ with } C_q = (e^2/h)\,\mathrm{Tr}\left\{\mathcal{Q}\right\} \qquad (2)$$

$$R_q = \frac{h}{2e^2}\frac{\mathrm{Tr}\left\{\mathcal{Q}^2\right\}}{\left(\mathrm{Tr}\left\{\mathcal{Q}\right\}\right)^2} \qquad (3)$$

Wigner-Smith time-delay matrix :

$$\mathcal{Q} = -\mathrm{i}\hbar\mathcal{S}^\dagger\frac{\partial\mathcal{S}}{\partial\varepsilon} \qquad (4)$$

In a chaotic cavity with perfect coupling ($\langle\mathcal{S}\rangle = 0$) :

$\mathcal{Q}^{-1}$ belongs to the Laguerre ensemble

Brouwer, Frahm & Beenakker, PRL (1997)

$$\mathcal{P}(\gamma_1,\cdots,\gamma_N) \propto \prod_{i<j}|\gamma_i-\gamma_j|^\beta\prod_{k=1}^{N}\gamma_k^{\beta N/2}e^{-\beta\gamma_k/2} \qquad (5)$$

Eigenvalue of $\mathcal{Q}$: proper time $\tau_i = \tau_{\mathrm{H}}/\gamma_i$ ($\tau_{\mathrm{H}} = h/\Delta$)

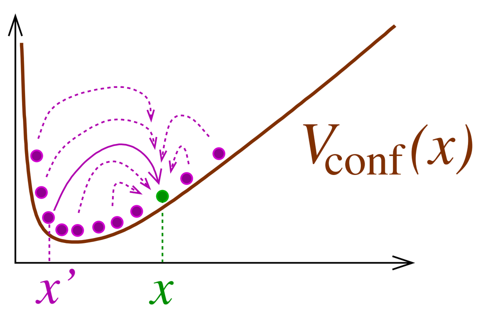
Question :

Statistical properties of $R_q = \frac{h}{2e^2}\frac{\sum_i\gamma_i^{-2}}{\left(\sum_i\gamma_i^{-1}\right)^2}$?

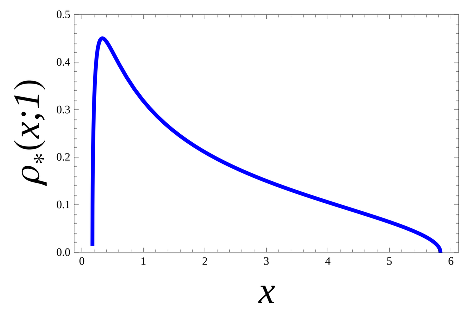
Coulomb gas method

Density of eigenvalues $\rho(x) = (1/N)\sum_i\delta(x-\gamma_i/N)$ has weight $\mathcal{P}(\{\gamma_i\}) \propto \exp\left\{-\frac{\beta}{2}N^2\mathscr{E}[\rho]\right\}$ where

$$\mathscr{E}[\rho] = \int_0^\infty \mathrm{d}x\,\rho(x)\,(x-\ln x) - \int_0^\infty \mathrm{d}x\mathrm{d}x'\,\rho(x)\,\rho(x')\,\ln|x-x'| \qquad (6)$$



$\Rightarrow$



Marčenko-Pastur law : (equilibrium density)

$$\rho_{\mathrm{MP}}(x) = \frac{\sqrt{(x_+-x)(x-x_-)}}{2\pi x}, \qquad x_\pm = 3\pm2\sqrt{2}$$

Coul. gas under constraints

Introduce $s = \mathrm{Tr}\left\{\mathcal{Q}\right\}/\tau_{\mathrm{H}} = (1/N)\sum_i x_i^{-1}$ and $r = N\,\mathrm{Tr}\left\{\mathcal{Q}^2\right\}/\tau_{\mathrm{H}}^2 = (1/N)\sum_i x_i^{-2}$

$$C_q = \frac{e^2}{\Delta}s \quad\text{and}\quad R_q = \frac{h}{Ne^2}\frac{r}{2s^2}. \qquad (7)$$

The joint distribution $P_N(s,r)$ is dominated by $\rho(x)$ minimizing $\mathscr{E}[\rho]$ under three constraints

$\Rightarrow$ consider $\mathscr{F}[\rho] = \mathscr{E}[\rho] + \mu_0\left(\int\mathrm{d}x\,\rho(x)-1\right) + \mu_1\left(\int\mathrm{d}x\,\rho(x)/x - s\right) + \mu_2\left(\int\mathrm{d}x\,\rho(x)/x^2 - r\right)$.

"Equilibrium" condition ($\delta\mathscr{F}/\delta\rho(x) = 0$) :

$$1 - \frac{1}{x} - \frac{\mu_1}{x^2} - \frac{2\mu_2}{x^3} = 2\fint_a^b \mathrm{d}x'\,\frac{\rho(x')}{x-x'} \quad \text{for } x\in[a,b] \qquad (8)$$

Large deviation function Ψ

Solution of (8) : $\rho(x)$ function of μ_1 & μ_2

constraints $\begin{cases} s = \int\mathrm{d}x\,\rho(x)/x \\ r = \int\mathrm{d}x\,\rho(x)/x^2\end{cases} \longrightarrow \begin{cases}\mu_1 = \mu_1^\star(s,r) \\ \mu_2 = \mu_2^\star(s,r)\end{cases}$

$\to$ get $\rho_\star(x;s,r)$ minimizing $\mathscr{F}[\rho]$

Joint distribution:

$$P_N(s,r) \underset{N\to\infty}{\sim} \exp\left\{-(\beta/2)N^2\Psi(s,r)\right\}, \qquad (9)$$

where

$$\Psi(s,r) = \mathscr{E}[\rho_\star(x;s,r)] - \mathscr{E}[\rho_{\mathrm{MP}}(x)] \qquad (10)$$

($\rho_{\mathrm{MP}}(x)$: solution for $\mu_1 = \mu_2 = 0$)

Thermodynamic identities

$$\frac{\partial\mathscr{E}[\rho_\star(x;s,r)]}{\partial s} = -\mu_1^\star(s,r), \qquad (11)$$

$$\frac{\partial\mathscr{E}[\rho_\star(x;s,r)]}{\partial r} = -\mu_2^\star(s,r). \qquad (12)$$

Typical fluctuations

$$\begin{cases} s_* = \int\rho_{\mathrm{MP}}(x)/x = 1 & \to \langle C_q\rangle = e^2/\Delta \\ r_* = \int\rho_{\mathrm{MP}}(x)/x^2 = 2 & \to \langle R_q\rangle \simeq h/(Ne^2)\end{cases} \qquad (13)$$

Expansion :
$\mathscr{E}[\rho_\star] \simeq \mathscr{E}[\rho_{\mathrm{MP}}] - (1/2)(s-1)^2\partial\mu_1/\partial s|_{\mathrm{MP}} - (1/2)(r-2)^2\partial\mu_2/\partial r|_{\mathrm{MP}} - (s-1)(r-2)\partial\mu_1/\partial r|_{\mathrm{MP}}$

$$\Psi(s,r) \simeq \begin{pmatrix}\delta s & \delta r\end{pmatrix}\begin{pmatrix}4 & 24\\ 24 & 160\end{pmatrix}^{-1}\begin{pmatrix}\delta s\\ \delta r\end{pmatrix}. \qquad (14)$$

$$\frac{\langle\delta C_q^2\rangle}{\langle C_q\rangle^2} \simeq \frac{\langle\delta R_q^2\rangle}{2\langle R_q\rangle^2} \simeq \frac{4}{\beta N^2}, \qquad \frac{\langle\delta C_q\delta R_q\rangle}{\sqrt{\langle\delta C_q^2\rangle\langle\delta R_q^2\rangle}} = +\frac{1}{\sqrt{2}}$$

Compact phase

Solution of (8) :

$$\rho_\star(x;s,r) = \frac{x^2+c\,x+d}{2\pi x^3}\sqrt{(b-x)(x-a)}, \qquad (15)$$

where $c = \mu_1/\sqrt{ab} + \mu_2\,(a+b)/(ab)^{3/2}$ and $d = 2\mu_2/\sqrt{ab}$. Given s and r, we obtain $[a,b]$ from

$$s = \left[2u\,(3-4u+18u^2-4u^3+3u^4) - v\,(1-u)^4(1+4u+u^2)\right]/(32u^3v), \qquad (16)$$

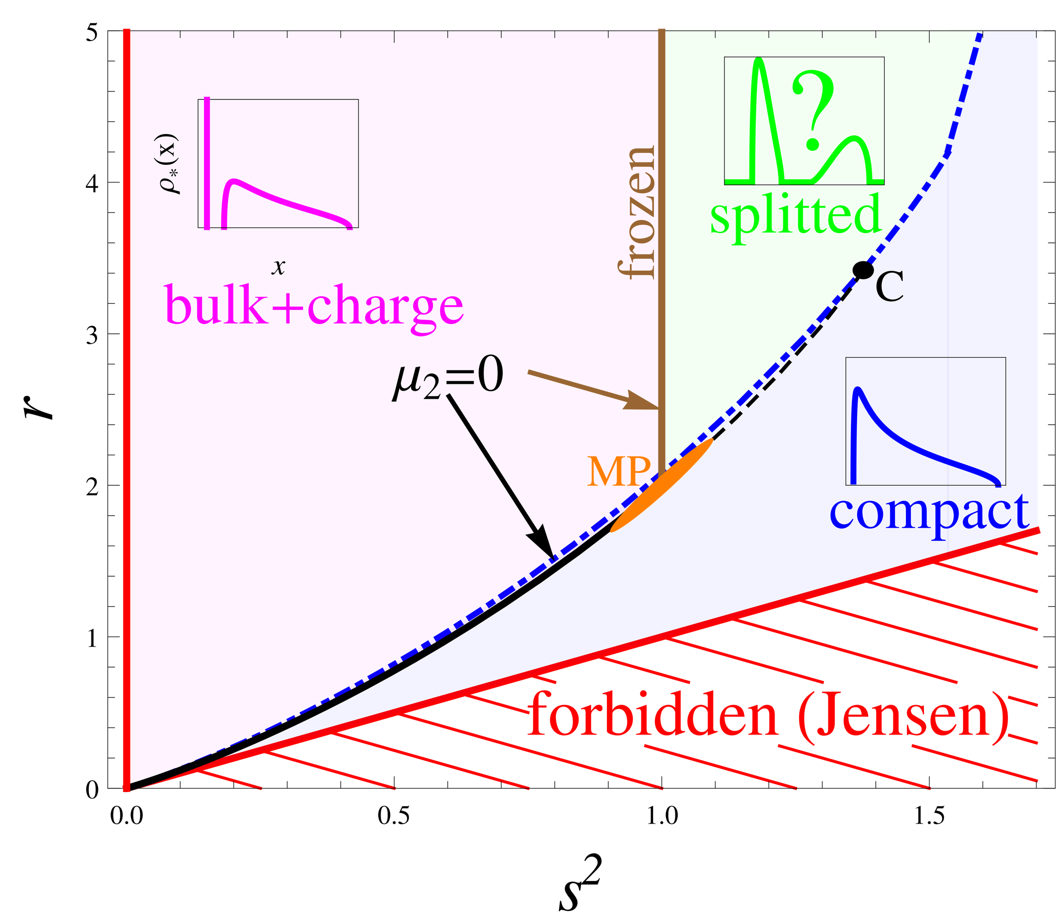
$$r = \left[2u\,(9-10u+39u^2-12u^3+39u^4-10u^5+9u^6) - 3v\,(1-u^2)^4\right]/(128u^4v^2), \qquad (17)$$

where $u = \sqrt{a/b}$ and $v = \sqrt{ab}$. Then the Lagrange multipliers are given by

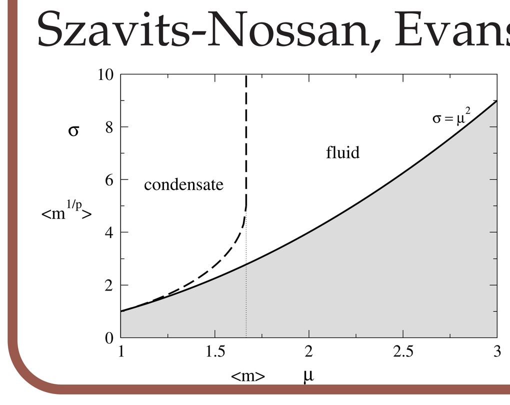
$$\mu_1 = v\left[2u\,(-9+4u-6u^2+4u^3-9u^4) + 3v\,(1-u^2)^2(1+u^2)\right]/(2u(1-u^2)^2), \qquad (18)$$

$$\mu_2 = -v^2\left[2u\,(-3+2u-3u^2) + v\,(1-u)^2\right]/(1-u^2)^2. \qquad (19)$$

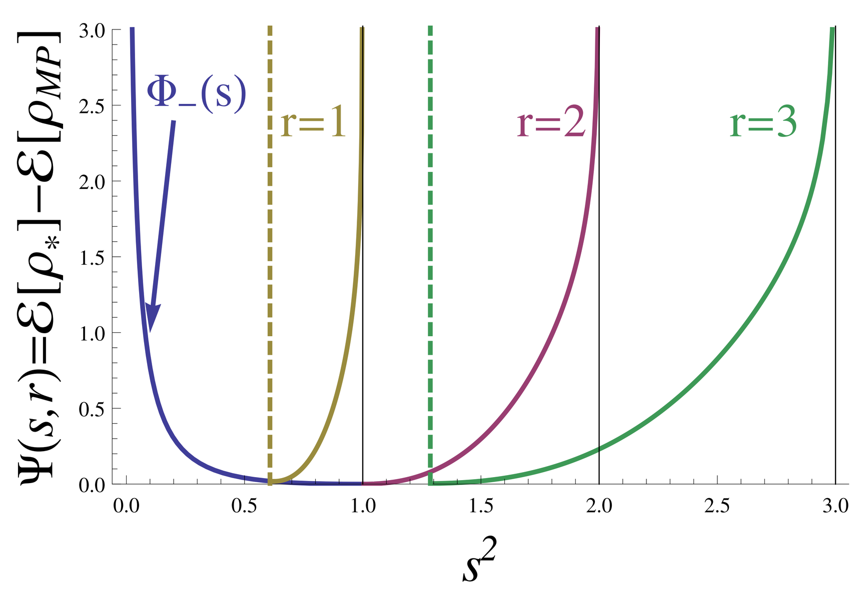
Phase diagram



Remark : case of *uncorrelated* variables: Szavits-Nossan, Evans & Majumdar, J.Phys.A (2014)



Energy of the gas



$$\Psi(s,r) \simeq -\frac{1}{2}\ln(r-s^2) \quad \text{for } r-s^2\to0^+ \qquad (20)$$

$P_N(R_q) \sim [R_q - h/(2Ne^2)]^{\beta N^2/4}$ for $R_q \to h/(2Ne^2)$.

Phase transition

$V_{\mathrm{eff}}(x) = x - \ln x + \mu_1/x + \mu_2/x^2 \Rightarrow \mu_2 < 0$: **instability**

$\Downarrow$

Phase transition

$\rho(x) = (1/N)\,\delta(x-x_1) + \tilde\rho(x)$ for $\mu_2 = 0$

"condensate"

$$r = \frac{1}{Nx_1^2} + \underbrace{\int\frac{\mathrm{d}x}{x^2}\,\tilde\rho(x)}_{\rho_\star(x;s)} \Rightarrow x_1 \simeq \frac{1}{\sqrt{N(r-r(s))}}$$

Energy : $\mathscr{E}[\rho_\star] \simeq \overbrace{\mathscr{E}[\rho_{\mathrm{MP}}] + \Phi_-(s)}^{\mathscr{E}[\rho_\star(x;s)]\text{, cf. [1]}} - (1/N)\ln x_1$

$$P_N(s,r) \sim (r-r(s))^{-\beta N/4}\exp\left\{-(\beta/2)N^2\Phi_-(s)\right\}. \qquad (21)$$

References

[1] Christophe Texier and Satya N. Majumdar, Phys. Rev. Lett. **110**, 250602 (2013).

[2] Aurélien Grabsch and Christophe Texier, preprint cond-mat arXiv:1407.3302.