UNIVERSITY OF HELSINKI
FACULTY OF SCIENCE

SPHERE PACKING BOUNDS AND RESTRICTED PARTITION FUNCTIONS

Lu Wei

Department of Mathematics and Statistics
University of Helsinki, Finland
lu.wei@helsinki.fi

SPHERE PACKING BOUNDS

We study sphere packing bounds in Stiefel manifold (unitary group) and Grassmann manifold. Packing bound is a basic coding-theoretic quantity that relates the code cardinality to the minimum distance of codewords. Besides coding-theoretic interest, codes in manifolds arise in MIMO wireless communications for transmission of information over noisy channels.

Evaluation of Gilbert–Varshamov lower bound or Hamming upper bound on the size of codes $|\mathcal{C}|$ amounts to calculate the volume of a normalized metric ball $\mu(B(r))$ in the manifold of interest

$$\underbrace{\frac{1}{\mu(B(r))}}_{\text{Gilbert–Varshamov bound}} \leq |\mathcal{C}| \leq \underbrace{\frac{1}{\mu(B(r/2))}}_{\text{Hamming bound}}. \tag{1}$$

In the unitary group, under Frobenius norm topological metric the ball is defined as

$$B(r) = \{\mathbf{U} \in U(n) \mid \underbrace{\|\mathbf{U} - \mathbf{I}_n\|_F \leq r}_{\text{topological metric}}\}. \tag{2}$$

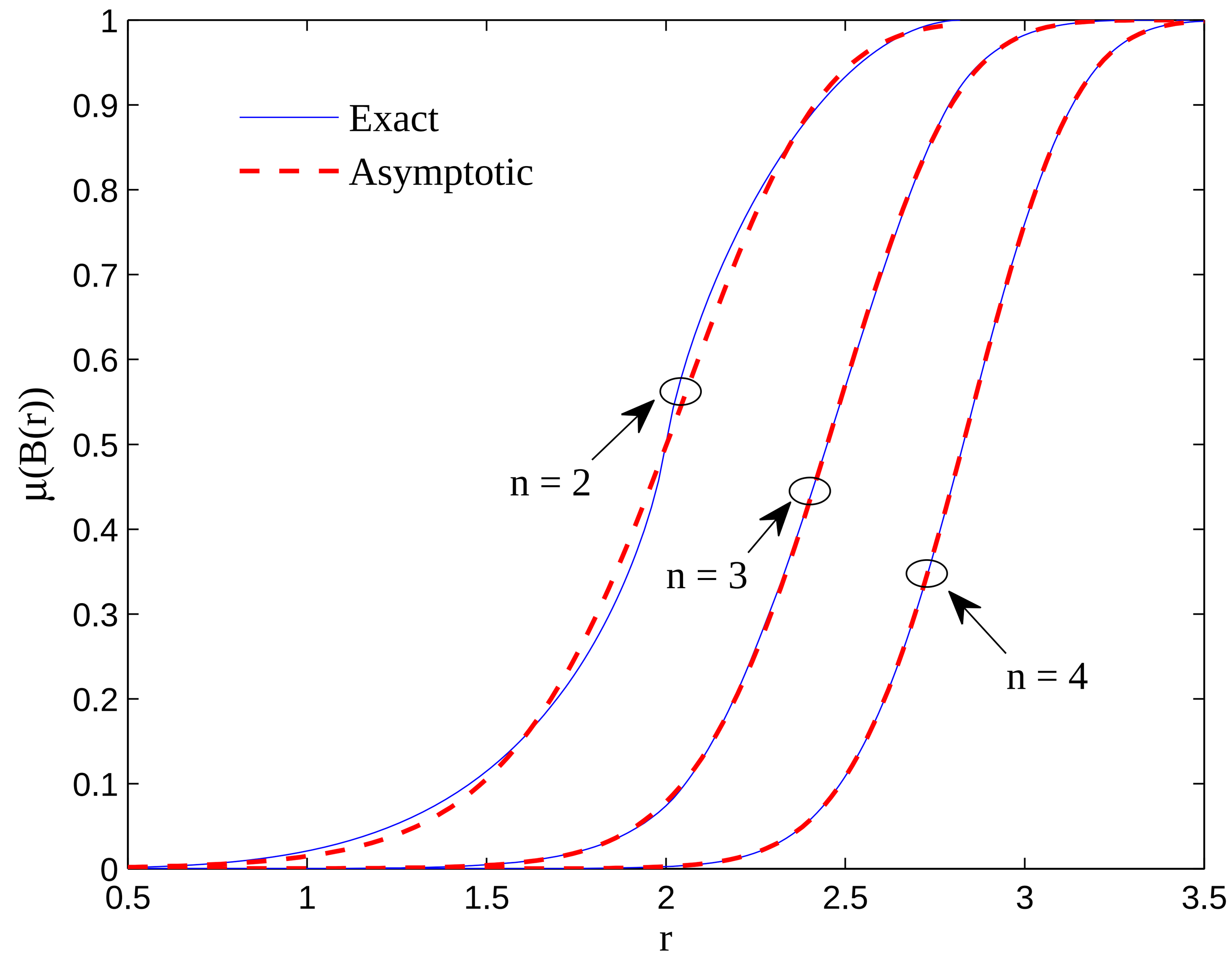

Figure 1: Volumes of metric ball: exact versus asymptotic (5).

RESTRICTED PARTITION FUNCTIONS

The volume of the metric ball (2) is given by [1]

$$\mu(B(r)) = \int_{\substack{-\pi\leq\theta_i\leq\pi,\\ \sum_{i=1}^{n}\sin^2\frac{\theta_i}{2}\leq\frac{r^2}{4}}} p(\theta_1,\ldots,\theta_n)\prod_{j=1}^{n}\mathrm{d}\theta_j, \tag{3}$$

- $p(\theta_1,\ldots,\theta_n) = c^{-1}\prod_{1\leq j<k\leq n}\left|e^{\imath\theta_j}-e^{\imath\theta_k}\right|^2$, Circular Unitary Ensemble
- a restricted partition function of CUE when $0 \leq r \leq 2\sqrt{n}$
- topological metric specifies the restriction

The restriction can be imposed in the integrand as

$$\begin{aligned}\mu(B(r)) &= \frac{1}{c}\int_0^{\frac{r^2}{4}}\int_{-\pi\leq\theta_i\leq\pi}\delta\left(t-\sum_{i=1}^{n}\sin^2\frac{\theta_i}{2}\right)\prod_{1\leq j<k\leq n}\left|e^{\imath\theta_j}-e^{\imath\theta_k}\right|^2\prod_{j=1}^{n}\mathrm{d}\theta_j\,\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\imath\left(1-e^{\imath\frac{r^2}{4}\nu}\right)}{\nu e^{\imath\frac{n}{2}\nu}}D_n(\nu)\,\mathrm{d}\nu,\end{aligned}$$

where

$$\begin{aligned}D_n(\nu) &= \frac{1}{c}\int_{-\pi\leq\theta_i\leq\pi}\prod_{1\leq j<k\leq n}\left|e^{\imath\theta_j}-e^{\imath\theta_k}\right|^2\prod_{j=1}^{n}e^{\imath\nu\frac{\cos\theta_j}{2}}\mathrm{d}\theta_j\\ &= \det\left(J_{j-k}\left(\frac{\nu}{2}\right)\right).\end{aligned}$$

Here $J_k(x)$ denotes Bessel function of the first kind.

ASYMPTOTIC BEHAVIOR

We claim that

$$\lim_{n\to\infty} D_n(\nu) = e^{-\frac{\nu^2}{16}}. \tag{4}$$

Two distinct proofs:

- matrix-variate hypergeometric functions and the associated zonal polynomials [2]
- Szegő's strong limit theorem for Toeplitz determinants [3]

Thus, an asymptotic volume formula is obtained as

$$\mu(B(r)) = \frac{1}{2}\mathrm{erf}(n) - \frac{1}{2}\mathrm{erf}\left(n-\frac{r^2}{2}\right), \quad n\to\infty, \tag{5}$$

where $\mathrm{erf}(x)$ is Gauss error function.

A BY-PRODUCT

An asymptotic formula for Toeplitz determinant of Bessel functions

$$\lim_{n\to\infty}\det\begin{pmatrix}J_0(x) & J_{-1}(x) & \cdots & J_{-n+1}(x)\\ J_1(x) & J_0(x) & \cdots & J_{-n+2}(x)\\ \vdots & \vdots & \ddots & \vdots\\ J_{n-1}(x) & J_{n-2}(x) & \cdots & J_0(x)\end{pmatrix} = e^{-\frac{x^2}{4}}. \tag{6}$$

Without tools from matrix-variate hypergeometric functions or Szegő's strong limit theorem, it is challenging to give an elementary proof of (6).

RESULTS FOR GRASSMANN MANIFOLD

In the complex Grassmann manifold, under chordal distance topological metric the volume of the ball equals [4]

$$\mu(B(r)) = \frac{1}{c'}\int_{\substack{0\leq x_i\leq 1,\\ \sum_{i=1}^{p}x_i\leq r}}\prod_{1\leq j<k\leq p}(x_j-x_k)^2\prod_{j=1}^{p}x_j^{n-2p}\,\mathrm{d}x_j, \tag{7}$$

which is a restricted partition function of Jacobi ensemble.

As p, $n\to\infty$ with $n-2p$ fixed

- $\mu(B(r)) = \frac{1}{2}\mathrm{erf}(\sqrt{2}n/2) - \frac{1}{2}\mathrm{erf}\left(2\sqrt{2}(n/4-r)\right)$
- proof: imposing the restriction in the integrand with a delta function, and the asymptotics of Hankel determinants [5]
- other topological metrics can be equally considered

REFERENCES

- [1] G. Han and J. Rosenthal, *IEEE Trans. Inf. Theory*, 52, (2006).
- [2] D. St. P. Richards, *Symmetry*, 3, (2011).
- [3] K. Johansson, *Bull. Sci. Math. (2)*, 112, (1988).
- [4] A. Barg and D. Y. Nogin, *IEEE Trans. Inf. Theory*, 48, (2002).
- [5] K. Johansson, *Ann. Math.*, 145, (1997).

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