Some results on Gaussian beta ensembles at high temperature

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1. Gaussian beta ensembles

Gaussian beta ensembles (GβE). Ensembles of real points with joint density

$$(\lambda_1, \dots, \lambda_n) \propto \prod_{i<j} |\lambda_j - \lambda_i|^\beta e^{-\frac{1}{2}(\lambda_1^2 + \dots + \lambda_n^2)} d\lambda.$$

They are generalizations of GOE, GUE and GSE, and can also be viewed as the equilibrium meas. of a one dim. Coulomb log-gas at the inverse temperature β.

Jacobi matrix models (Dumitriu and Edelman 2002).

$$T_{n,\beta} = \begin{pmatrix} \mathcal{N}(0,1) & \tilde\chi_{(n-1)\beta} & & \\ \tilde\chi_{(n-1)\beta} & \mathcal{N}(0,1) & \tilde\chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \tilde\chi_\beta & \mathcal{N}(0,1) \end{pmatrix},$$

where $\begin{cases} \mathcal{N}(\mu, \sigma^2): \text{ Gaussian distribution with mean } \mu \text{ and variance } \sigma^2, \\ \tilde\chi_k = \frac{1}{\sqrt{2}}\chi_k = \sqrt{\text{Gamma}(\frac{k}{2}, 1)}. \end{cases}$

- The eigenvalues $(\lambda_1, \dots, \lambda_n)$ of $T_{n,\beta}$ are distributed as GβE.
- Let $q_j = |v_j(1)|$, where $v_1, \dots, v_n$ are the corresponding normalized eigenvectors. Then $(q_1^2, \dots, q_n^2)$ have Dirichlet distribution with parameter $\frac{\beta}{2}$, and are independent of $(\lambda_1, \dots, \lambda_n)$.

We are interested in the following quantities.

1. Empirical distribution/measure: $L_{n,\beta} = \frac{1}{n}\sum_{j=1}^n \delta_{\lambda_j}$.
2. Spectral measure (to be defined on the right): $\mu_{n,\beta} = \sum_{j=1}^n q_j^2 \delta_{\lambda_j}$.
3. Bulk statistics: $\xi_n = \sum_{j=1}^n \delta_{n(\lambda_j - E)}$, for fixed E.

2. Spectral measures of Jacobi matrices

Jacobi matrices (finite and infinite):

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \\ & & b_{n-1} & a_n \end{pmatrix}, \quad J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & b_2 & a_3 & b_3 \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \quad \text{where } \begin{cases} a_i \in \mathbb{R}, \\ b_i > 0. \end{cases}$$

Definition (Spectral measure)
There is a probability measure μ on $\mathbb{R}$ satisfying

$$\langle \mu, x^k \rangle := \int x^k \mu(dx) = (J^k e_1, e_1) = J^k(1,1), \quad k = 0, 1, \dots.$$

The measure μ is called the spectral measure of J in case it is unique.

- Finite Jacobi matrices of order n are one to one correspondence with probability measures supported on n points. In fact,

$$\mu = \sum_{j=1}^n q_j^2 \delta_{\lambda_j}, \quad q_j = |v_j(1)|, \text{where } \begin{cases} \lambda_1, \dots, \lambda_n: \text{ e.values of } J, \\ v_1, \dots, v_n: \text{ normalized e.vectors.} \end{cases}$$

- Given a nontrivial probability measure μ with all finite moments, constructing orthogonal polynomials in $L^2(\mu)$ leads to Jacobi coef. $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$.

Example

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \end{pmatrix} \leftrightarrow sc(dx) = \frac{1}{2\pi}\sqrt{4 - x^2}dx: \text{ semi-circle distribution;}$$

$$\begin{pmatrix} 0 & \sqrt{1} & & \\ \sqrt{1} & 0 & \sqrt{2} & \\ & \ddots & \ddots & \ddots \end{pmatrix} \leftrightarrow \mu(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx: \text{ standard Gaussian distribution.}$$

3. GβE for fixed β

For fixed β, as $n \to \infty$,

$$\frac{\sqrt{2}}{\sqrt{n\beta}} T_{n,\beta} = \frac{\sqrt{2}}{\sqrt{n\beta}} \begin{pmatrix} \mathcal{N}(0,1) & \tilde\chi_{(n-1)\beta} & & \\ \tilde\chi_{(n-1)\beta} & \mathcal{N}(0,1) & \tilde\chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \tilde\chi_\beta & \mathcal{N}(0,1) \end{pmatrix} \overset{a.s.}{\to} \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 0 & 1 \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

which implies Wigner's semi-circle law. Let $\tilde L_{n,\beta}$ and $\tilde\mu_{n,\beta}$ be the empirical measure and the spectral measure of $\frac{\sqrt{2}}{\sqrt{n\beta}} T_{n,\beta}$, respectively.

Theorem (Wigner's semi-circle law (Johansson 1998, Dumitriu and Edelman 2006, D. 2016))
As $n \to \infty$,

$$\tilde L_{n,\beta} \text{ and } \tilde\mu_{n,\beta} \to sc \text{ (almost surely),}$$

this means that for any bounded continuous function f,

$$\langle \tilde L_{n,\beta}, f \rangle = \frac{1}{n}\sum_{j=1}^n f(\lambda_j) \to \langle sc, f \rangle \text{ almost surely,}$$

and $\langle \tilde\mu_{n,\beta}, f \rangle \to \langle sc, f \rangle$ almost surely.

- *For empirical measures,*

$$n\Big(\langle \tilde L_{n,\beta}, f \rangle - \mathbb{E}[\langle \tilde L_{n,\beta}, f \rangle]\Big) \overset{d}{\to} \mathcal{N}(0, \hat\sigma_f^2),$$

with explicit formula of $\hat\sigma_f^2$.

- *For spectral measures,*

$$\sqrt{\frac{n\beta}{2}}\Big(\langle \tilde\mu_{n,\beta}, f \rangle - \mathbb{E}[\langle \tilde\mu_{n,\beta}, f \rangle]\Big) \overset{d}{\to} \mathcal{N}\Big(0, \langle sc, f^2 \rangle - (\langle sc, f \rangle)^2\Big).$$

The CLTs hold for f having continuous derivative of polynomial growth.

Theorem (Bulk statistics (Valkó and Virág 2009))
For $E \in (-2, 2)$, $\tilde\xi_n$ converges weakly to the so-called Sine$_\beta$ point process.

It was shown that the Sine$_\beta$ point process converges to a homogeneous Poisson point process as $\beta \to 0$ (Allez and Dumaz 2014).

4. GβE at high temperature

As $n \to \infty$ with $n\beta = 2\alpha$,

$$\begin{pmatrix} \mathcal{N}(0,1) & \tilde\chi_{(n-1)\beta} & & \\ \tilde\chi_{(n-1)\beta} & \mathcal{N}(0,1) & \tilde\chi_{(n-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \tilde\chi_\beta & \mathcal{N}(0,1) \end{pmatrix} \overset{d}{\to} \begin{pmatrix} \mathcal{N}(0,1) & \tilde\chi_{2\alpha} & & \\ \tilde\chi_{2\alpha} & \mathcal{N}(0,1) & \tilde\chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \end{pmatrix} =: J_\alpha.$$

→ $\mu_{n,\beta}$ converges in distribution to μ_α, the spectral measure of J_α;
→ the means $\bar L_{n,\beta} = \bar\mu_{n,\beta}$ converge to $\bar\mu_\alpha$, the mean of μ_α (D. and Shirai 2015).

The measure $\bar\mu_\alpha$ is called the prob. meas. of associated Hermite polynomials

$$\begin{pmatrix} 0 & \sqrt{\alpha+1} & & \\ \sqrt{\alpha+1} & 0 & \sqrt{\alpha+2} & \\ & \ddots & \ddots & \ddots \end{pmatrix} \leftrightarrow \bar\mu_\alpha = \frac{e^{-\frac{E^2}{2}}}{\sqrt{2\pi}} \frac{1}{|\hat f_\alpha(E)|^2},$$

where $\hat f_\alpha(E) = \sqrt{\frac{\alpha}{\Gamma(\alpha)}} \int_0^\infty t^{\alpha-1} e^{-\frac{t^2}{2} + iEt} dt$.

Theorem (Benaych-Georges and Péché 2015)
As $n \to \infty$ with $n\beta = 2\alpha$, $L_{n,\beta} \to \bar\mu_\alpha$ (in probability).

Theorem (Central limit theorem (D. and Nakano 2016))
As $n \to \infty$ with $n\beta = 2\alpha$,

$$\sqrt{n}\Big(\langle L_{n,\beta}, f \rangle - \mathbb{E}[\langle L_{n,\beta}, f \rangle]\Big) \overset{d}{\to} \mathcal{N}(0, \sigma_f^2),$$

provided that f has continuous derivative of polynomial growth.

Theorem (Poisson statistics (D. and Nakano 2016))
As $n \to \infty$ with $n\beta = 2\alpha$, the point process

$$\xi_n = \sum_{j=1}^n \delta_{n(\lambda_j - E)}$$

converges weakly to a homogeneous Poisson point process with intensity $\bar\mu_\alpha(E)$.

The Poisson statistics was first proved by Benaych-Georges and Péché by analyzing the joint density function. However, it was unable to show that the intensity is $\bar\mu_\alpha(E)$. We refine Minami's method and can show the right intensity.

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