From non-ergodic eigenvectors to local resolvent statistics and back: a random matrix perspective

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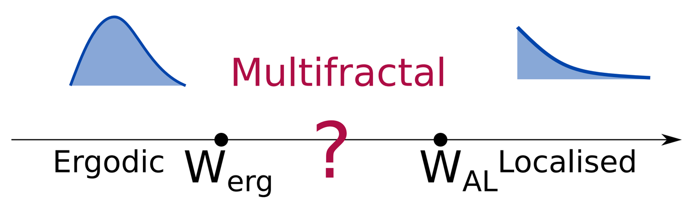
The phenomenon of many-body localisation (MBL) has attracted considerable interest in recent years in the study of non-equilibrium quantum sytems. A feature that is not well understood is the existence of an intermediate extended non-ergodic "bad metal" phase between the localised and extended ones. To help the debate about the existence and role of extended, non-ergodic phases in MBL and the related problem of Anderson localisation on the Bethe lattice, we study such a phase in the simpler setting of the generalised Rosenzweig-Porter random matrix model. We confirm previous results about the multifractality of the eigenstates, and show that the intermediate phase can be characterised studying the local resolvent in a non-standard scaling limit.



Generalised Rosenzweig-Porter model

Random matrix model with the $N \times N$ Hamiltonian

$$H = A + \frac{\mu}{N^{\gamma/2}} V \tag{1}$$

where A is diagonal with real i.i.d. entries $a_i \sim p_A(a_i)$, and V belongs to the Gaussian unitary ensemble (GUE) with variance 1. The parameter γ is a proxy for the disorder strength. All quantities are considered in the large N limit.

Level statistics: transition between Poisson ($\gamma > 2$) and Wigner-Dyson ($\gamma < 2$). Well known [1] and was interpreted as the fingerprint of a transition between a localised and an ergodic phase.

Eigenstates: it was argued recently [2] that the they are actually delocalised only for $\gamma \leq 1$. There is a third, intermediate phase $\gamma \in (1, 2)$ in which the eigenstates are multifractals extended over $\propto N^{2-\gamma}$ sites.

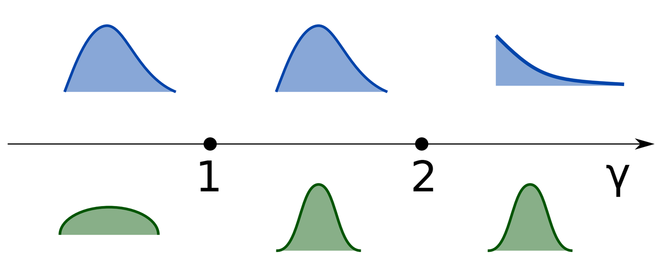


Fig. 1: Level statistics and density of states in the three phases.

The model undergoes two transitions, AL-like at $\gamma = 2$ and ergodic at $\gamma = 1$.

Recurrence relation for the local resolvent

Resolvent matrix:

$$\mathbf{G}(z) = (z - H)^{-1} \tag{2}$$

From block matrix inversion formula, equivalence between RV

$$G_{00}^{(N+1)}(z)^{-1} \stackrel{d}{=} z - H_{00} - \frac{\mu^2}{N^\gamma} \sum_{ij} G_{ij}^{(N)}(z) V_{0i} V_{j0} \ . \tag{3}$$

Correlations between terms in the sum, and their variances are small for large N (shown perturbatively). The sum converges to a non-fluctuating quantity involving $\overline{G}(z) = \mathrm{Tr}\mathbf{G}(z)/N$. All diagonal elements of $\mathbf{G}$ are equivalent. Therefore the distribution of the local resolvent is given by

$$G_{ii}(z)^{-1} \stackrel{d}{=} z - a_i - \frac{\mu^2}{N^{\gamma-1}} \overline{G}(z) - \frac{\mu}{N^{\gamma/2}} V_{ii} \ . \tag{4}$$

The values $\gamma = 1$ and $\gamma = 2$, corresponding to the transitions, appear naturally when comparing the order in N of the terms in the expression. In the intermediate region $\gamma \in (1, 2)$, the last term can be neglected.

Local resolvent statistics

We consider the local resolvent at $z = \lambda - \mathrm{i}\eta$, with λ in the bulk of the spectrum.
The statistics of $\mathrm{Im}G_{ii}$, with η small but finite in the thermodynamics limit, has been studied in many disordered systems, and is widely used to characterise localisation transitions [3].
Using eq. (4) we derived the probability distribution for the real and imaginary parts of the local resolvent, both with η small but fixed, and $\eta \approx N^{-\delta}$ (Fig. 2).
For the imaginary part we obtained, with η of order one,

$$p_{\mathrm{Im}G}(x) = \phi_A(x) \frac{\sqrt{\eta}}{2x^{3/2}\sqrt{1 - \eta x}} \tag{5}$$

with support $(0, 1/\eta)$ and where $\phi_A(x)$ depends weakly on x.

The details depend on the distribution p_A through ϕ_A, but some features are universal: a peak at $\mathrm{Im}G_{typ} \approx \eta$, the cutoff at $1/\eta$ and a power law $\propto x^{-3/2}$ in between. This behaviour is typical of localised systems [3].

Taking a scaling $\eta \approx N^{-\delta}$ we found two different regimes, separated by a critical $\eta_c = N^{1-\gamma}$.

- If $\delta > \delta_c = \gamma - 1, \eta \gg \eta_c$, the behaviour is the same as for the previous case.
- If $\delta < \delta_c, \eta \ll \eta_c$, the distribution has the same form with $\eta \to \mu^2 N^{1-\gamma} \mathrm{Im}\overline{G}$

In the first case, the PDF looks as if the system were in a localised phase. In the second case, $\mathrm{Im}G_{typ} \gg \eta$ and the system looks as in a delocalised phase.

This suggests the idea of a localisation length in energy space η_c.

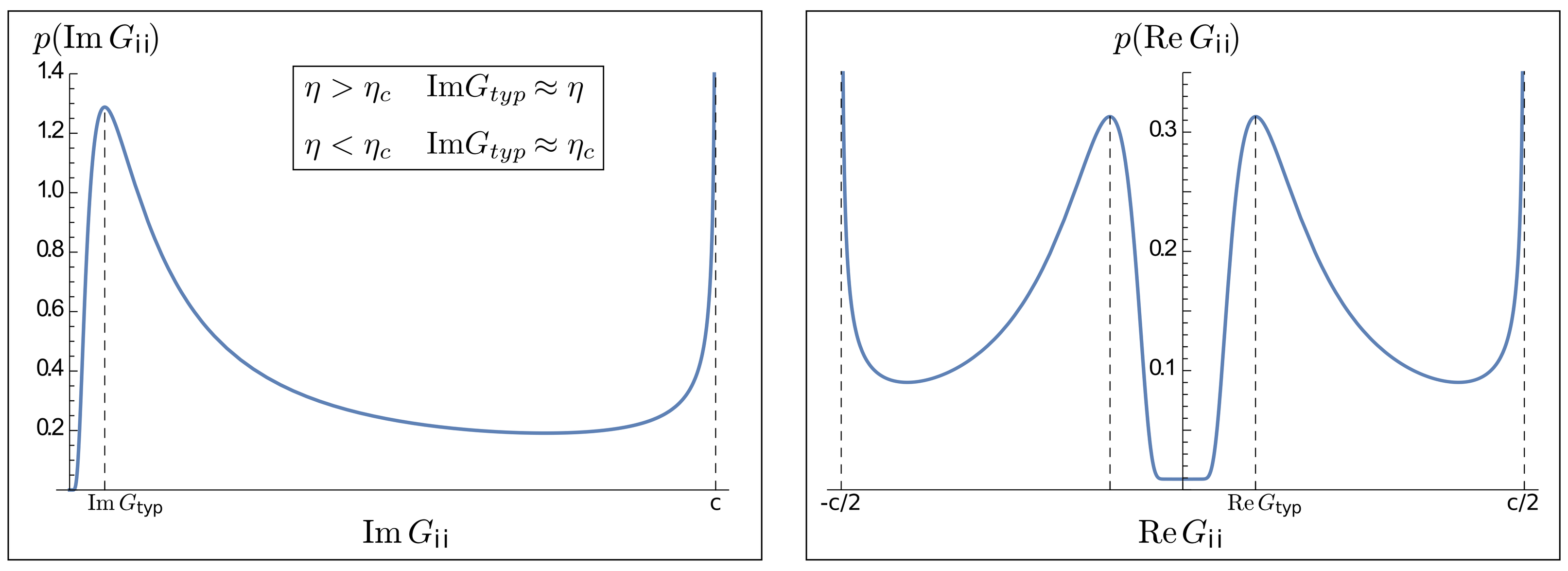


Fig. 2: Probability densities for real and imaginary parts of the local resolvent.

Dyson Brownian motion

Matrix-valued stochastic process

$$\mathrm{d}M_{ij}(t) = \sqrt{\frac{(1+\delta_{ij})\sigma^2}{2}} \mathrm{d}W_{ij}, \tag{6}$$

where W_{ij} are independent standard complex Wiener processes, with $W_{ij} = W_{ji}^*$. With initial conditions $M(0) = A$ and $\sigma^2 = N^{-\gamma}$, at $t = \mu^2$ $M(\mu^2) = H$ is a GRP matrix.

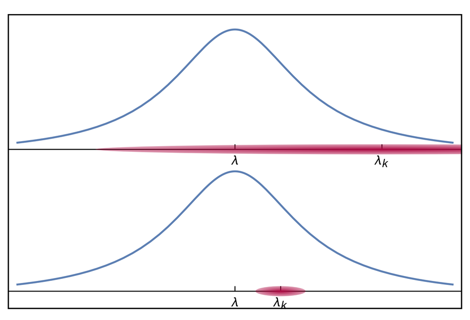
Using Itō's calculus we derived evolution equations for the eigenvalues, eigenvectors, and for the quantities, $u_{i|j} = [|\psi_j^{(i)}|^2]$ where $[\dots]$ indicates the average performed only over the off-diagonal elements V_{ij} [4, 5]. We analysed the evolution equation

$$\partial_t u_{i|j}(t) = N^{-\gamma} \sum_{k \neq i} \frac{u_{k|j}(t) - u_{i|j}(t)}{(\lambda_k - \lambda_i)^2} \ , \tag{7}$$

using an ansatz for $u_{i|j}$ inspired by the result on the local resolvent statistics, but without fixing the value of δ_c.
The ansatz is consistent if the eigenstates are delocalised over the $N^{2-\gamma}$ sites closest in energy, compatible with the results of Ref. [2].

Conclusions

- The extended, non-ergodic phase can be characterised by the statistics of the local resolvent $G_{ii}(\lambda - \mathrm{i}\eta)$ in the non-standard scaling limit $\eta \propto N^{-\delta}$.



- Insights from local resolvent statistics and Dyson Brownian confirm that for $\gamma \in (1, 2)$ the eigenstates of the GRP are extended over the $N^{2-\gamma}$ sites closest in energy.
- Local resolvent statistics explained intuitively as eigenvalue contribute to $\mathrm{Im}G_{ii}$ as "packets" of width $\eta_c = N^{1-\gamma}$.

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