Non-interacting fermions in a box and Random Matrix Theory

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Fermions as a determinantal point process

N non-interacting spinless fermions at $T=0$, in a general potential:

$$H=\sum_{i=1}^{N}-\frac{\hbar^2}{2m}\partial_{x_i}^2+V(x_i) \quad (1)$$

ψ_k is the single-particle wave-function of energy E_k

Harmonic potential $V(x)=V_0\left(\frac{x}{r_0}\right)^2$: positions $\leftrightarrow$ GUE eigenvalues

$$|\psi_{\rm GS}(x_1,\dots,x_N)|^2=\frac{1}{Z_N}\prod_{i<j=1}^{N}|x_i-x_j|^2\,e^{-\sum_{i=1}^{N}\left(\frac{x_i}{r_0}\right)^2} \quad (2)$$

General potential: still determinantal point process

$$|\psi_{\rm GS}(x_1,\dots,x_N)|^2=\frac{1}{N!}\det_{1\le i\le j\le N}K_N(x_i,x_j) \quad (3)$$

$$K_N(x,y)=\sum_{k=0}^{N-1}\psi_k^*(x)\psi_k(y) \quad (4)$$

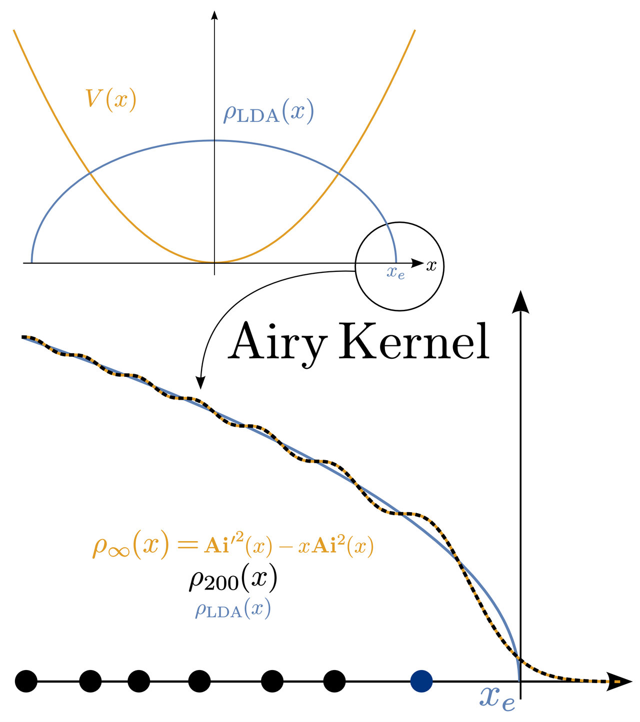
Local density approximation

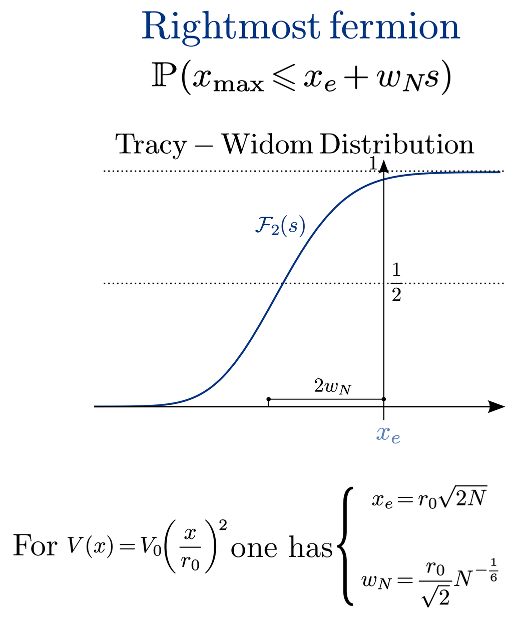
Semiclassical formula describing the density in the bulk

$$\rho_{\rm LDA}(x)=\frac{1}{N}\sqrt{\frac{2m}{\hbar^2\pi^2}}\theta[\mu-V(x)]\sqrt{\mu-V(x)} \quad (5)$$

This approximation breaks down at the edge x_e such that $V(x_e)=\mu$

For $V(x)=V_0\left|\frac{x}{r_0}\right|^p$, properties at the edge described by Airy kernel [1]





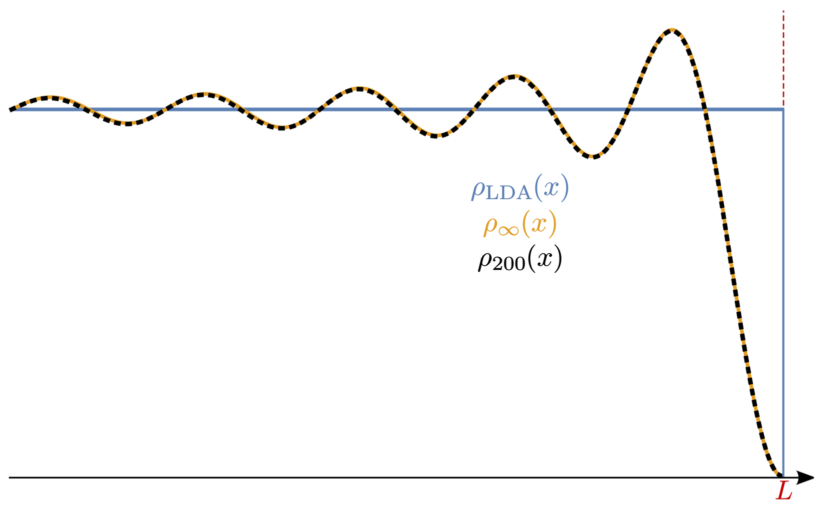
Hard wall potential

Hard wall potential $W(x)=\begin{cases}0, & -L<x<L\\ +\infty, & |x|>L\end{cases}$

Kernel close to the edge found in [2],[3]

$$\frac{2L}{N}K_N\left(L(1-\frac{2x}{N}),L(1-\frac{2y}{N})\right)\to K_{\rm wall}(x,y)=\frac{\sin(\pi(x-y))}{\pi(x-y)}-\frac{\sin(\pi(x+y))}{\pi(x+y)}$$

After a change of variable $x\to\sqrt{x}$, $K_{\rm wall}$ is the Bessel kernel of index $\alpha=\frac{1}{2}$

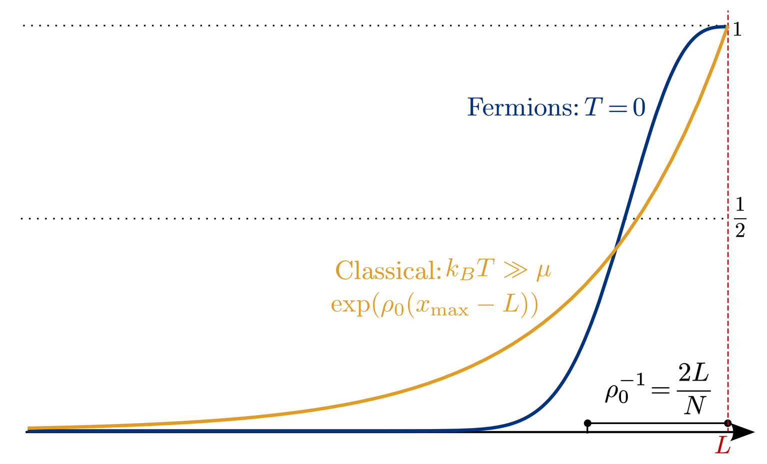


Rightmost fermion cumulative distribution function [3]

$$P\left(x_{\max}<L(1-\frac{2s}{N})\right)=\det(I-P_{[0,s]}K_{\rm wall}P_{[0,s]})=D_-(s)$$

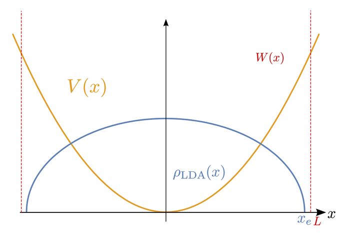
$$D_-(s)=\begin{cases}1-\frac{2\pi^2}{9}s^3+\frac{2\pi^4}{75}s^5+O(s^7), & s\ll 1\\ \exp(-\frac{\pi^2}{16}s^2+\frac{\pi}{4}s-\frac{1}{8}\log(\pi s)+O(1)), & s\gg 1\end{cases}$$

$D_-(s)$ is ALSO the probability that there is at most 1 eigenvalue in the interval $[0,s]$ for GOE



General soft potential + wall

Soft potential $V(x)$ + hard wall $W(x)$ close to the edge. [3]



$$w_NK_N(L+w_Nx,L+w_Ny)=\int_l^\infty\tau(x+z,z)\tau(y+z,z)dz$$

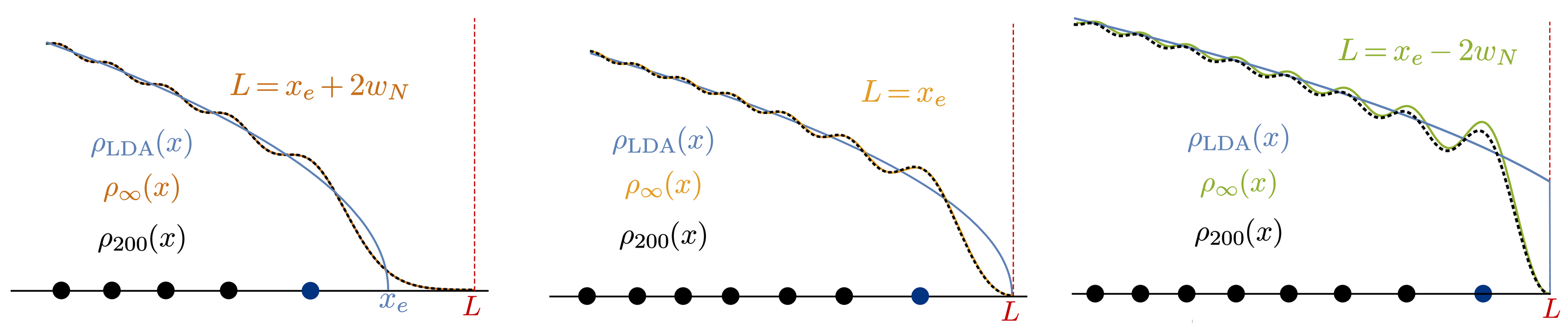
$$\tau(x,y)=\frac{\mathrm{Bi}(y)\,\mathrm{Ai}(x)-\mathrm{Ai}(y)\,\mathrm{Bi}(x)}{\sqrt{\mathrm{Bi}^2(y)+\mathrm{Ai}^2(y)}},\quad w_N=\left(\frac{\hbar^2}{2mV'(x_e)}\right)^{\frac{1}{3}},\quad l=\frac{L-x_e}{w_N}$$

This kernel interpolates between $K_{\rm Ai}$ and $K_{\rm wall}$

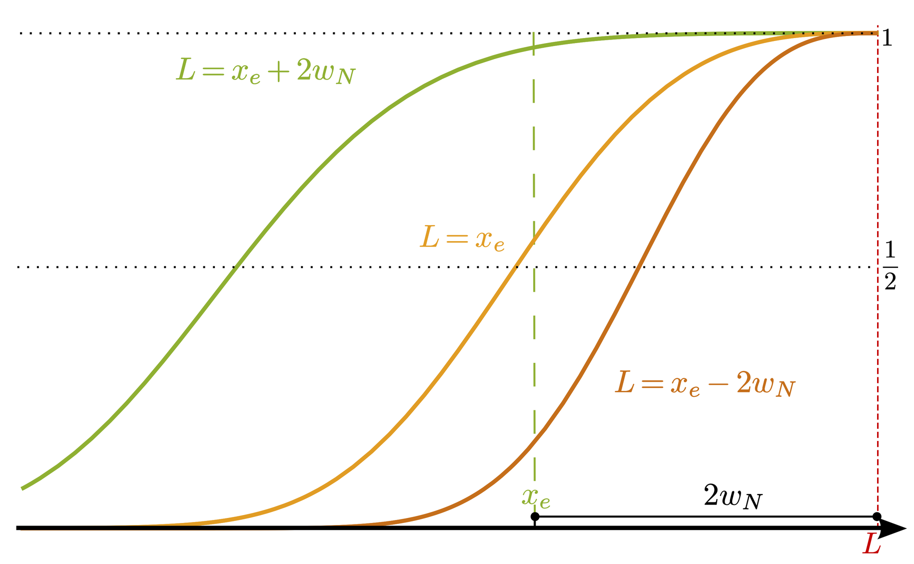
$$w_NK_N(x_e+w_Nx,x_e+w_Ny)\to K_{\rm Ai}(x,y),\ l\to+\infty \quad (6)$$

$$w_NK_N(L+w_Nx,L+w_Ny)\to\frac{\sqrt{|l|}}{\pi}K_{\rm wall}(\frac{\sqrt{|l|}}{\pi}x,\frac{\sqrt{|l|}}{\pi}y),\ l\to-\infty \quad (7)$$

Comparison with simulation for $N=200$ fermions



New rightmost fermion cumulative distribution function



References

[1] D. S. Dean, P. Le Doussal, S. N. Majumdar, and G. Schehr, Finite-Temperature Free Fermions and the Kardar-Parisi-Zhang Equation at Finite Time, Phys. Rev. Lett. **114**(11), 110402 (2015).

[2] P. Calabrese, M. Mintchev, and E. Vicari, The entanglement entropy of one-dimensional systems in continuous and homogeneous space, J. Stat. Mech. **2011**(09), P09028 (2011).

[3] B. Lacroix a chez toine,P. Le Doussal, S. N. Majumdar, and G. Schehr, Non-interacting fermions in a box and Random Matrix Theory, *in preparation*.