The Semi-Circle Law in Algebraic Random Matrix Theory

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Framework

Given a measure μ on a set X, we consider the algebra

$$L(X,\mu,\mathbb{F}) := \bigcap_{1\le p<\infty} L^p(X,\mu,\mathbb{F}),$$

where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. An $n\times n$ **algebraic random matrix** is an element of the algebra

$$M_n(\mathbb{F})\otimes L(X,\mu,\mathbb{F}) \cong M_n(L(X,\mu,\mathbb{F})).$$

The map

$$\operatorname{tr}\otimes\mathbb{E} : M_n(\mathbb{F})\otimes L(X,\mu,\mathbb{F}) \to \mathbb{F}$$

endows this algebra with the structure of a **non-commutative probability space** [1]. If for each $n\in\mathbb{N}$, (X_n,μ_n) is a measure space, a set $\{A(n)\}_{n\in\mathbb{N}}$ such that every $A(n)\in M_n(\mathbb{F})\otimes L(X_n,\mu_n,\mathbb{F})$, is called an **algebraic random matrix ensemble**.

Main interest

For an algebraic random matrix ensemble $\{A(n)\}_{n\in\mathbb{N}}$, we consider the numbers

$$\alpha_{m,n} := (\operatorname{tr}\otimes\mathbb{E})(A(n)^m),$$

which are called the **moments**. In the case of Hermitian matrices, these are precisely the moments of the discrete eigenvalue distributions. We are especially interested in the **limiting moments**

$$\alpha_m := \lim_{n\to\infty}\alpha_{m,n},$$

whenever they exist. In the case of Hermitian matrices, these correspond to the limiting eigenvalue distribution.

Primary example

The well-known **GUE-ensemble** can be either represented as a sequence of random variables on the sets of Hermitian $n\times n$-matrices, given by the probability distribution

$$\frac{1}{Z_n}\exp(-\frac{n}{2}\operatorname{Tr}(M^2))\,\mathrm{d}M,$$

where $\mathrm{d}M=\prod_{i=1}^n \mathrm{d}M_{ii}\prod_{i<j}\mathrm{d}\Re M_{ij}\,\mathrm{d}\Im M_{ij}$ is the flat Lebesgue measure, or as the matrices

$$A(n)=\frac{1}{\sqrt{2n}}\begin{pmatrix}\sqrt{2}e_{11} & e_{12}+ie_{21} & \cdots & e_{1n}+ie_{n1}\\ e_{12}-ie_{21} & \sqrt{2}e_{22} & \cdots & e_{2n}+ie_{n2}\\ \vdots & \vdots & & \vdots\\ e_{1n}-ie_{n1} & e_{2n}-ie_{n2} & \cdots & \sqrt{2}e_{nn}\end{pmatrix}$$

where the variables $\{e_{ij}\}_{i,j}$ are independent standard normal distributed random variables.

Generalization

An **S-ensemble** over $\mathbb{F}$ with coefficient $\gamma\in\mathbb{F}$ is an algebraic random matrix ensemble $\{A(n)\}_{n\in\mathbb{N}}$ such that:

- There is a sequence $\{k_n\}_{n\in\mathbb{N}}\subseteq\mathbb{N}$ such that
$$A(n)\in M_n(\mathbb{F})\otimes L(\mathbb{R}^{k_n},\mu_{k_n},\mathbb{F}),$$
where μ_k is the k-fold product $\mu\times\cdots\times\mu$ and μ is the standard normal distribution on $\mathbb{R}$.
- For every $n\in\mathbb{N}$ and $i,j\in\{1,\dots,n\}$, we have that
$$X(n)_{ij}=\sum_{r=1}^{k_n}\lambda_{ijr}(n)e_r,$$
with each $\lambda_{ijr}(n)\in\mathbb{F}$ and $e_r:\mathbb{R}^{k_n}\to\mathbb{R}$ is the r'th projection function.
- For every $n\in\mathbb{N}$ and $i\neq j$:
$$\sum_{r=1}^{k_n}\lambda_{ijr}(n)\lambda_{jir}(n)=\frac{\gamma}{n}$$
- There is a constant $C\ge 0$ such that for every $n\in\mathbb{N}$ and $i,j\in\{1,\dots,n\}$,
$$\left|\sum_{r=1}^{k_n}\lambda_{ijr}(n)^2\right|\le\frac{C}{n}$$
- For every $n\in\mathbb{N}$ and $i,j,l,m\in\{1,\dots,n\}$ such that $(i,j)\neq(l,m)\neq(j,i)$,
$$\sum_{r=1}^{k_n}\lambda_{ijr}(n)\lambda_{lmr}(n)=0$$

Main theorem

The limiting moments of an S-ensemble with coefficient γ are

$$\alpha_m=\begin{cases}0 & \text{if } m \text{ is odd}\\ \gamma^{m/2}C_{m/2} & \text{if } m \text{ is even}\end{cases}$$

where $C_k=\frac{1}{k+1}\binom{2k}{k}$ is the k'th Catalan number.

Interpretation and remarks

- For $\gamma=1$, these moments are precisely the moments of **Wigner's semi-circle law**, supported on $[-2,2]$, with weight function
$$w(x)=\frac{1}{2\pi}\sqrt{4-x^2}$$
with respect to the standard Lebesgue measure on $\mathbb{R}$.
- The class of S-ensembles also contains non-Hermitian random matrix ensembles.
- The class of S-ensembles has a rich algebraic structure.
- The proof is a generalization of the analysis of the GUE-ensemble in [1] and has the flavour of the discussion found in [2].

Bibliography

[1] A. Nica and R. Speicher, *Lectures on the combinatorics of Free Probability*. Cambridge University Press, 2006.

[2] G. Anderson, A. Guionnet, and O. Zeitouni, *An Introduction to Random Matrices*. Cambridge University Press, 2009.

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