Gap probabilities in piecewise thinned Airy and Bessel point processes

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Based on several works with T. Claeys and A. Doeraene

Gap in the piecewise thinned Airy point process

The Airy point process is a determinantal point process on $\mathbb{R}$, arising near soft edges of certain large random matrices.

Let $m \in \mathbb{N}_{>0}$, $\vec{s} = (s_1, ..., s_m) \in [0,1]^m$ and $\vec{x} = (x_1, ..., x_m) \in \mathbb{R}^m$ be such that $-\infty < x_m < ... < x_1 < x_0 = +\infty$.

For $j \in \{1, 2, ..., m\}$, each particle on the interval (x_j, x_{j-1}) is removed with probability s_j.

We consider the probability to observe a gap on $(x_m, +\infty)$ in the thinned process. This probability can be written as a Fredholm determinant:

$$F(\vec{x}; \vec{s}) = \det\left(1 - \chi_{(x_m,+\infty)} \sum_{j=1}^{m} (1 - s_j) \mathcal{K}^{\mathrm{Ai}} \chi_{(x_j, x_{j-1})}\right).$$

Exact expression for $F(\vec{x}; \vec{s})$ and a system of Painlevé II equations

If $m = 1$, Tracy and Widom ('94) have shown that $F(x, s)$ can be expressed in terms of a solution to a Painlevé II equation.

This result was generalised by Claeys-Doeraene ('18) for an arbitrary $m \geq 1$ as follows

$$F(\vec{x}, \vec{s}) = \prod_{j=1}^{m} \exp\Big(- \int_0^{+\infty} \xi q_j^2(\xi; \vec{x}, \vec{s}) d\xi \Big),$$

where q_1,...,q_m satisfy a system of m coupled Painlevé II equations

$$q_j'' = (\xi + x_j) q_j + 2 q_j \sum_{\ell=1}^{m} q_\ell^2, \qquad j = 1, ..., m,$$

$$q_j(\xi; \vec{x}, \vec{s}) = \sqrt{s_{j+1} - s_j}\, \mathrm{Ai}(\xi + x_j)(1 + o(1)), \qquad \text{as } \xi \to +\infty,$$

where $s_{m+1} := 1$.

Note that small gaps (i.e. asymptotics for $F(r\vec{x}, \vec{s})$ as $r \to +\infty$ and $x_m > 0$) can be deduced immediately from the above result.

Large gap asymptotics

Large gap asymptotics (i.e. asymptotics for $F(r\vec{x}, \vec{s})$ as $r \to +\infty$ and $x_1 < 0$) are much harder to obtain than small gap asymptotics.

For $m = 1$, large r asymptotics for $F(rx, 0)$ are known from Deift-Its-Krasovsky ('08) and Baik-Buckingham-Di Franco ('08).

Asymptotics for $F(rx, s)$ as $r \to +\infty$ with $s > 0$ were recently obtained by Bothner-Buckingham ('18).

These two results are generalized (joint work with T. Claeys '18) for an arbitrary m below

For $s_1, ..., s_m \in (0, 1]$, as $r \to +\infty$, we have

$$F(r\vec{x}, \vec{s}) = e^{-4\pi^2 \sum_{1 \leq k < j \leq m} \beta_j \beta_k \Sigma(x_k, x_j)} \prod_{j=1}^{m} F\Big(r x_j, \frac{s_j}{s_{j+1}}\Big)\Big(1 + \mathcal{O}\Big(\frac{\log r}{r^{3/2}}\Big)\Big),$$

where $s_{m+1} := 1$, $\beta_j := \frac{1}{2\pi i} \log \frac{s_{j+1}}{s_j}$ and $\Sigma(x_k, x_j) = \frac{1}{2\pi^2} \log \frac{\sqrt{x_k} + \sqrt{x_j}}{\sqrt{x_k} - \sqrt{x_j}}$.

For $s_1 = 0$ and $s_2, ..., s_m \in (0, 1]$, as $r \to +\infty$, we have

$$F(r\vec{x}, \vec{s}) = F(rx_1, 0) F(r\vec{y}, \vec{s}_0) \prod_{j=2}^{m} \left[\left(\frac{2(x_1 - x_j)}{x_1 - 2x_j} \right)^{\beta_j^2} e^{-2i\beta_j r^{3/2} |x_1| \, |x_1 - x_j|^{1/2}} \right] \Big(1 + \Big(\frac{\log r}{r^{3/2}}\Big)\Big),$$

where

$$\vec{y} = (x_2 - x_1, ..., x_m - x_1) \quad \text{and} \quad \vec{s}_0 = (s_2, ..., s_m).$$

Gap in the piecewise thinned Bessel point process

The Bessel point process is a determinantal point process on $\mathbb{R}^+$, arising near hard edges of certain large random matrices, and depends on a parameter $\alpha > -1$.

Let $m \in \mathbb{N}_{>0}$, $\vec{s} = (s_1, ..., s_m) \in [0,1]^m$ and $\vec{x} = (x_1, ..., x_m) \in (\mathbb{R}^+)^m$ be such that $0 =: x_0 < x_1 < x_2 < ... < x_m < +\infty$.

For $j \in \{1, 2, ..., m\}$, each particle on the interval (x_{j-1}, x_j) is removed with probability s_j.

We study the probability to observe a gap on $(0, x_m)$ in the thinned process, which can be written as a Fredholm determinant:

$$F(\vec{x}, \vec{s}) = \det\left(1 - \chi_{(0,x_m)} \sum_{j=1}^{m} (1 - s_j) \mathcal{K}^{\mathrm{Be}} \chi_{(x_{j-1}, x_j)}\right),$$

Exact expression for $F(\vec{x}; \vec{s})$ and a system of Painlevé V equations

If $m = 1$, Tracy and Widom ('94) have shown that $F(x, s)$ can be expressed in terms of a solution to a Painlevé V equation.

This result was generalised by C-Doeraene ('18) for an arbitrary $m \geq 1$ as follows

$$F(r\vec{x}, \vec{s}) = \prod_{j=1}^{m} \exp\left(-\frac{x_j}{4} \int_0^r \log\left(\frac{r}{\xi}\right) q_j^2(\xi; \vec{x}, \vec{s}) d\xi \right), \qquad r > 0,$$

where q_1,...,q_m satisfy a system of m coupled Painlevé V equations

$$\xi q_j \Big(1 - \sum_{\ell=1}^{m} q_\ell^2\Big) \sum_{\ell=1}^{m} (\xi q_\ell q_\ell')' + \xi \Big(1 - \sum_{\ell=1}^{m} q_\ell^2\Big)^2 \Big((\xi q_j')' + \frac{x_j q_j}{4}\Big) + \xi^2 q_j \Big(\sum_{\ell=1}^{m} q_\ell q_\ell'\Big)^2 = \alpha^2 \frac{q_j}{4},$$

$$q_j(\xi; \vec{x}, \vec{s}) = \sqrt{s_{j+1} - s_j}\, J_\alpha(\sqrt{x_j \xi})(1 + \mathcal{O}(\xi)), \qquad \text{as } \xi \to 0, \quad j = 1, ..., m.$$

Note that small gaps (i.e. asymptotics for $F(r\vec{x}, \vec{s})$ as $r \to 0$) can be deduced immediately from the above result.

Large gap asymptotics

Large gap asymptotics (i.e. asymptotics for $F(r\vec{x}, \vec{s})$ as $r \to +\infty$) are much harder to obtain than small gap asymptotics.

For $m = 1$, large r asymptotics for $F(rx, 0)$ are known from Deift-Krasovsky-Vasilevska ('11).

Asymptotics for $F(rx, s)$ as $r \to +\infty$ and $s > 0$ were not known.

Asymptotics in the above two regimes are generalized (C '18) for an arbitrary m below.

For $s_1, ..., s_m \in (0, 1]$, as $r \to +\infty$, we have

$$F(r\vec{x}, \vec{s}) = e^{-4\pi^2 \sum_{1 \leq k < j \leq m} \beta_j \beta_k \Sigma(x_k, x_j)} \prod_{j=1}^{m} F\Big(r x_j, \frac{s_j}{s_{j+1}}\Big)\Big(1 + \mathcal{O}\Big(\frac{\log r}{\sqrt{r}}\Big)\Big),$$

with an explicit asymptotics for $F(rx, s)$ ($m = 1$), and $s_{m+1} := 1$.

For $s_1 = 0$ and $s_2, ..., s_m \in (0, 1]$, as $r \to +\infty$, we have

$$F(r\vec{x}, \vec{s}) = F(rx_1, 0) F(r\vec{y}, \vec{s}_0) \Big(1 + \Big(\frac{\log r}{\sqrt{r}}\Big)\Big),$$

where

$$\vec{y} = (x_2 - x_1, ..., x_m - x_1) \quad \text{and} \quad \vec{s}_0 = (s_2, ..., s_m).$$