Non-Hermitian ensembles and Painlevé critical asymptotics

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Normal matrix model

We are interested in the normal random matrix model defined by

$$dP_N(z_1, z_2, \ldots, z_N; t) = \frac{1}{\mathcal{Z}_N(t)} |\Delta(z)|^2 \prod_{j=1}^{N} e^{-NV_t^{(s)}(z_j)}\, dA(z_j),$$

with $z_j \in \mathbb{C}$ and potential

$$V_t^{(s)}(z) = |z|^{2s} - t(z^s + \overline{z}^s), \qquad s \in \mathbb{N}.$$

The eigenvalues $z_1, \ldots, z_N$ display an interesting behaviour:

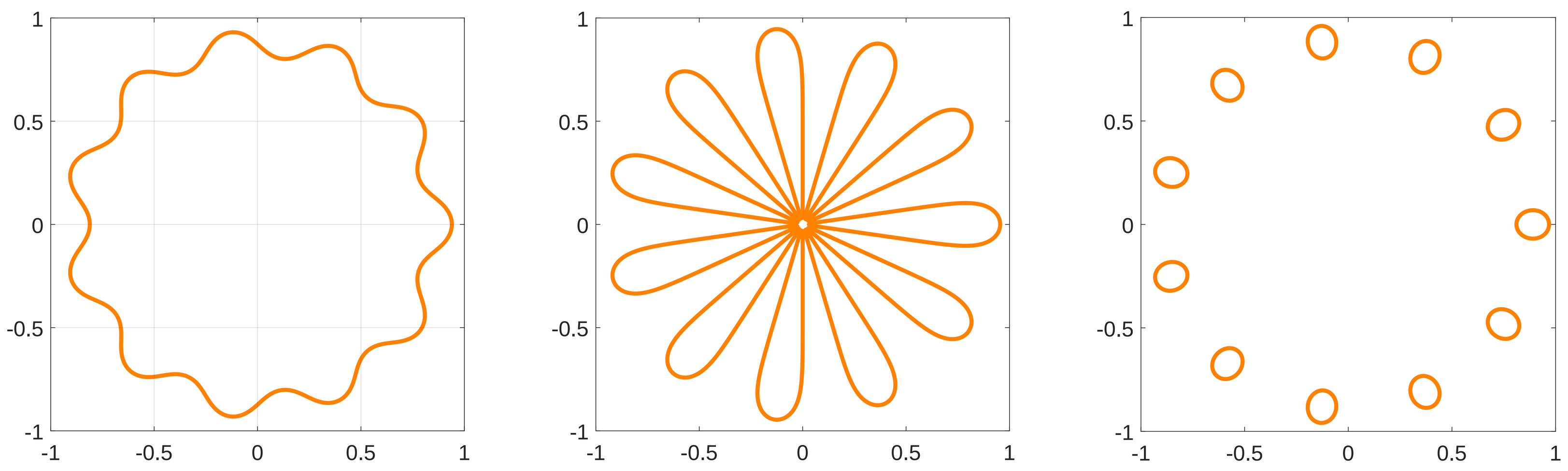

Figure 1: The limiting eigenvalue distribution is supported on the interior of the orange curves. Here $s = 11$ and $t = t_c - 0.1$ (left), $t = t_c$ (centre) and $t = t_c + 0.1$ (right). At the special value $t = t_c$, the support becomes disconnected.

In this poster our goal is to investigate the partition function $\mathcal{Z}_N(t)$ near the critical value $t = t_c = 1/\sqrt{s}$.

Reduction to the Ginibre ensemble

The first observation is that we can use symmetry to write $\mathcal{Z}_N(t)$ as an average over the Ginibre ensemble:

$$\mathcal{Z}_{Ns}(t) = C_{N,s} \prod_{l=0}^{s-1} \mathcal{Z}_N^{(\gamma_l)}(x), \qquad \gamma_l := -2\left(1 - \frac{l+1}{s}\right),$$

where

$$\mathcal{Z}_N^{(\gamma)}(x) = \int_{\mathbb{C}^N} |\Delta(\vec{z})|^2 \prod_{j=1}^{N} |z_j - x|^{\gamma} e^{-N|z_j|^2} dA(z_j), \qquad x := t\sqrt{s}.$$

Criticality now corresponds to the spectral variable x colliding with the boundary of the circular law (*i.e.* $|x| = 1$). When $|x| < 1 - \delta$ (sub-critical), the asymptotics were obtained in [2].

Painlevé and non-Hermitian matrix integrals

Our main result for finite N characterizes the partition function as a solution of the σ-form of Painlevé V.

Theorem 1. *The 'reduced' partition functions $\mathcal{Z}_N^{(\gamma)}(x)$ can be written as*

1. *An average over the CUE:*

$$\mathcal{Z}_N^{(\gamma)}(x) = c_{N,\gamma} \left\langle \prod_{j=1}^{N} e^{-\frac{i\gamma\theta_j}{4}} |1 + e^{i\theta_j}|^{\frac{\gamma}{2}} e^{Nx^2 e^{i\theta_j}} \right\rangle_{\mathrm{CUE}}.$$

2. *The σ-form of Painlevé V:*

$$\mathcal{Z}_N^{(\gamma)}(x) = c_{N,\gamma} \exp\left(\int_0^{Nx^2} \frac{y_N(t) + \frac{\gamma N}{2}}{t}\, dt\right)$$

where $y_N(t) \equiv \sigma(t)$ satisfies the equation

$$(t\sigma'')^2 - [\sigma - t\sigma' + 2(\sigma')^2 + (N - \gamma)\sigma']^2 + 4\sigma'(\sigma' - \tfrac{\gamma}{2})^2(\sigma' + N) = 0, \quad (1)$$

with initial condition

$$\sigma(t) \sim -\frac{\gamma N}{2} + \frac{t\gamma}{2}, \qquad t \to 0.$$

The first part above can be arrived at by a judicious inspection of formulas in [2]. Then the second part is a consequence of the first due to results of Forrester and Witte '02.

Large N asymptotic results

Asymptotic results for related orthogonal polynomials have been studied in various works, but the critical case only very recently in [1]. For the partition function, we obtain:

Theorem 2. *If $\gamma = 2k$, where $k \in \mathbb{N}$, then for*

$$|x|^2 = 1 - \frac{u}{\sqrt{N}}, \qquad u \in \mathbb{R}$$

we have the following asymptotics:

$$\frac{\mathcal{Z}_N^{(2k)}(x)}{E_{N,k}} = \exp\left(-\sqrt{N}ku - \int_u^{\infty} v(\xi)\, d\xi\right)(1 + o(1)), \quad N \to \infty,$$

uniformly for u in compact subsets of $\mathbb{R}$, where $E_{N,k}$ is a completely explicit pre-factor. The function v satisfies the σ-form of the Painlevé IV equation:

$$(v'')^2 + 4(v')^2(v' + k) - (sv' - v)^2 = 0, \qquad (2)$$

subject to the boundary condition

$$v(s) = -ks - \frac{k}{s} + O(s^{-3}), \qquad s \to -\infty.$$

We believe this result persists to non-integer k, indeed a naive rescaling of equation (1) reproduces exactly the Painlevé IV in (2). The advantage of integer k is the *duality* (Forrester and Rains '08):

$$\frac{\mathcal{Z}_N^{(2k)}(x)}{\widetilde{E}_{N,k}} = |x|^{2Nk + 2k^2} \int_{[0,\infty)^k} \prod_{j=1}^{k} e^{-\sqrt{N}|x|^2 r_j} \left(1 + \frac{r_j}{\sqrt{N}}\right)^N |\Delta(\vec{r})|^2 d\vec{r}, \quad (3)$$

making $N \to \infty$ asymptotics easy to compute. For k not integer, we use Riemann–Hilbert techniques.

Extensions of the theory: Truncated unitary matrices

Let U be an $M \times M$ matrix from the CUE and decompose

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A is an $N \times N$ sub-block. As shown in [3], the joint density of eigenvalues for A is proportional to

$$\prod_{j=1}^{N} (1 - |z_j|^2)^{M-N-1} |\Delta(\vec{z})|^2, \qquad |z_j| \leq 1.$$

If we now consider averaged powers of the characteristic polynomial of A:

$$\mathcal{Z}_{N,\mathrm{TCUE}}^{(\gamma)}(x) := \int_{\mathbb{D}^N} |\Delta(\vec{z})|^2 \prod_{j=1}^{N} |z_j - x|^{\gamma} (1 - |z_j|^2)^{M-N-1} dA(z_j)$$

Then we show:

- $\mathcal{Z}_{N,\mathrm{TCUE}}^{(\gamma)}(x)$ is expressed similarly, but now in terms of *Painlevé VI*.
- For $\gamma = 2k$, $\mathcal{Z}_{N,\mathrm{TCUE}}^{(\gamma)}(x)$ satisfies a similar duality as (3).
- In the regime of *weak non-unitarity* where $M - N$ is fixed and $x^2 = 1 - u/N$, the boundary limit involves *Painlevé V*.
- In the regime $M = \alpha N$, the limit at the boundary is described in terms of *Painlevé IV*, like in the Ginibre case.

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References

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