Eigenvector Correlations in Quaternionic Ginibre Ensembles

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Introduction

As is widely known, non-Hermitian matrices feature distinct left and right eigenvectors, neither of which forms an orthonormal system. It was suggested by Chalker and Mehlig to study the correlation of these eigenvectors through the expectation values of their overlaps, and they demonstrated an approach via Schur decomposition for complex Ginibre ensembles.[1] Moreover, recent contributions for real and complex Ginibre ensembles have been made by Fyodorov,[2] and Bourgade and Dubach.[3] A newly published pre-print by Dubach treats the quaternionic case as well.[4]

We explore eigenvector correlations for matrices with quaternion Gaussian entries following Chalker's and Mehlig's idea. This allows us to derive expectations for finite N of both diagonal and off-diagonal overlaps, and to motivate heuristically their asymptotic behaviour, where we consider eigenvalues in the bulk of the spectrum and close to zero.

Quaternions and Ginibre ensembles

- Instead of working with matrices with quaternion entries, we use the standard complex embedding $\chi(A) \in M_{2N}(\mathbb{C})$ of $A \in M_N(\mathbb{H})$ induced by

$$\chi\colon \mathbb{H} \to M_2(\mathbb{C}) \tag{1}$$

$$a + \hat{\imath}a_1 + \hat{\jmath}b_1 + \hat{k}b_2 \mapsto \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \quad \text{with } a = a_1 + \mathrm{i}a_2,\ b = b_1 + \mathrm{i}b_2 \in \mathbb{C}\,.$$

  We call matrices with such substructure *quaternionic*.
- Quaternionic Ginibre ensembles are a familiy of quaternionic random matrices with iid Gaussian entries. A Ginibre ensemble is a random matrix distributed by the pdf

$$P(G)\mathrm{d}[G] := \left(\frac{1}{\pi\sigma^2}\right)^{2N^2} \exp\left[-\frac{1}{\sigma^2}\mathrm{Tr}\, GG^\dagger\right] \prod_{i,j=1}^{2N} \mathrm{dRe}G_{ij}\mathrm{dIm}G_{ij}. \tag{2}$$

- The eigenvalues of a Ginibre matrix are non-degenerate (up to a set with measure zero) and appear in conjugate pairs.

Overlap matrix and Schur decomposition

- For a square matrix G with right eigenvectors R_i and left eigenvectors L_i (associated with eigenvalues λ_i), we define the *overlap matrix*

$$\mathcal{O}_{ij} := L_i^\dagger \cdot L_j\, R_j^\dagger \cdot R_i. \tag{3}$$

- For Hermitian matrices, the overlaps form the unity matrix if eigenvectors are properly normalised. Generally, the overlap matrix is non-trivial.
- The overlaps matrix entries are invariant under unitary transformations of the eigenvalues.
- A $2N \times 2N$ quaternionic matrix G can be expressed as its Schur normal form

$$U^\dagger G U = \widetilde{G} = \begin{pmatrix} Z_1 & & T \\ & Z_2 & \\ & & \ddots & \\ 0 & & & Z_N \end{pmatrix} \quad \text{with blocks } Z_i = \begin{pmatrix} z_i & 0 \\ 0 & \overline{z_i} \end{pmatrix} \tag{4}$$

  and symplectic and unitary $U \in \mathrm{USp}(2N)/\mathrm{U}(1)^N$. The Z_i contain the standard complex eigenvalue pairs z_i and $\overline{z_i}$ of G.
- Under the decomposition (4), the pdf (2) transforms to

$$P\,(G)\,\mathrm{d}\,[G] = C_N \prod_{1\le i<j\le N} |z_i - z_j|^2 |z_i - \overline{z_j}|^2 \prod_{i=1}^{N} |z_i - \overline{z_i}|^2 \tag{5}$$

$$\times \exp\left[-\frac{2}{\sigma^2}\sum_{k=1}^{N} |z_k|^2 - \frac{1}{\sigma^2}\mathrm{Tr}\,(T^\dagger T)\right] \mathrm{d}[z]\mathrm{d}[T]\mathrm{d}[S]\,.$$

  Here, C_N is a normalisation constant, and $\mathrm{d}[z] = \prod_{i=1}^{N} \mathrm{d}^2 z_i$ and the measure $\mathrm{d}[T]$ are defined as in (2) over all real and imaginary parts of the matrix elements $T_{2i-1,2j+1}$ and $T_{2i-1,2j+2}$ for $i, j = 1, 2, \ldots, N-1$.
- G and $\widetilde{G}$ have the same overlaps.
- The indexing of R_i, L_i and λ_i is arbitrary, and we can focus on $\mathcal{O}_{11}$ and $\mathcal{O}_{12}$.

Result: One-point correlation function and limiting distribution

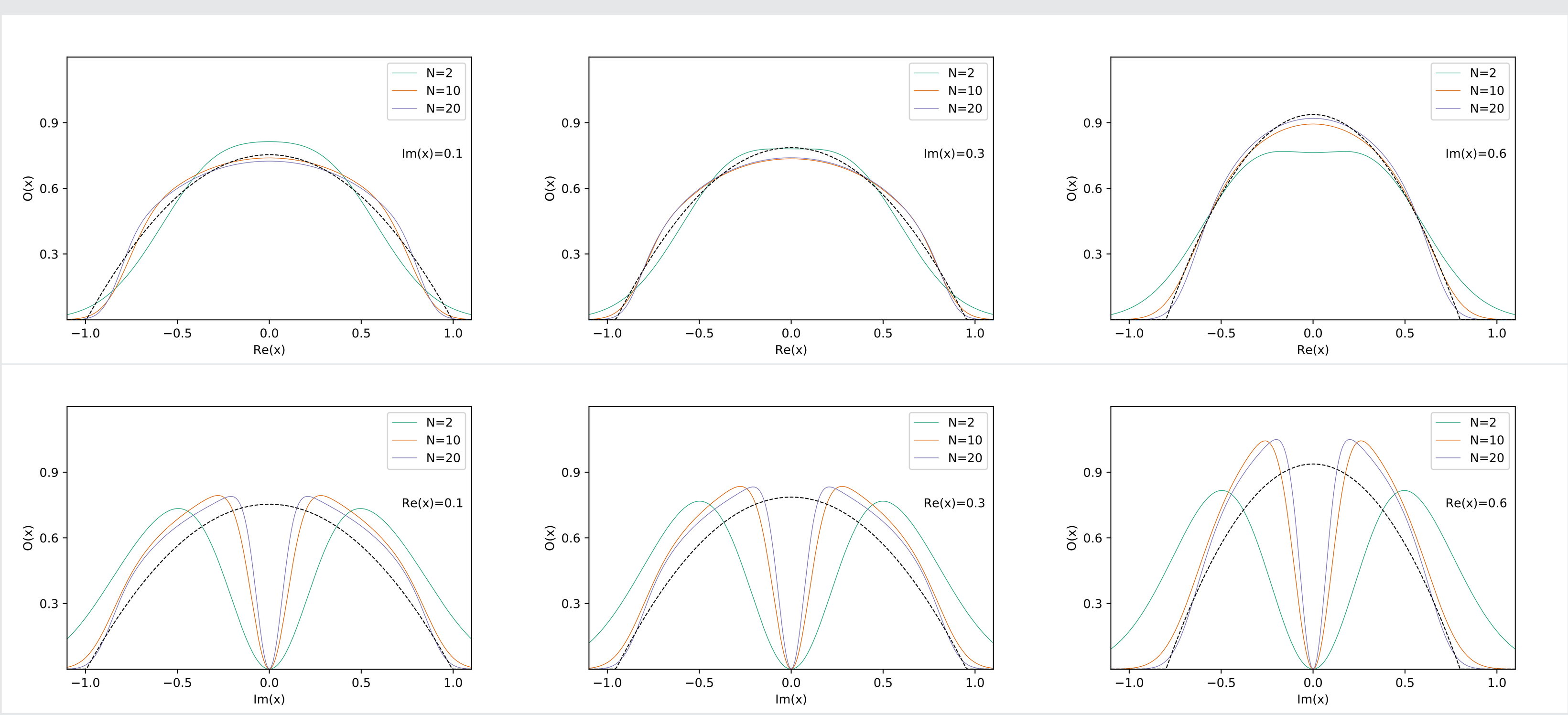

Figure 1: Projections of $\mathcal{O}(x)$ for varying N according to Eq. (8), with Re(x) (first row) or Im(x) (second row) fixed; rescaled by $\sigma^2 = \frac{8}{\pi^2 N}$. Pfaffians were computed using PFAPACK.[5] Limiting distribution (dashed line) according to Eq. (9).

Finite N results

- The overlap factors can be expressed as

$$\mathcal{O}_{11} = \sum_{k=1}^{N} \left(|b_k|^2 + |b'_k|^2\right), \quad \mathcal{O}_{12} = -\overline{b_2} \sum_{k=2}^{N} \left(b_k \overline{d_k} + b'_k \overline{d'_k}\right), \tag{6}$$

  where b_p, b'_p, and d_p and d'_p depend recursively on the entries of $\widetilde{G}$.
- Averaging over T, we find closed formulae

$$\langle \mathcal{O}_{11} \rangle_T = \prod_{l=2}^{N} \left(1 + \frac{\sigma^2}{4|z_1 - z_l|^2} + \frac{\sigma^2}{4|z_1 - \overline{z_l}|^2}\right), \tag{7}$$

$$\langle \mathcal{O}_{12} \rangle_T = -\frac{\sigma^2}{4|z_1 - z_2|^2} \prod_{l=3}^{N} \left(1 + \frac{\sigma^2}{4(z_1 - z_l)(\overline{z_2} - \overline{z_l})} + \frac{\sigma^2}{4(z_1 - \overline{z_l})(\overline{z_2} - z_l)}\right).$$

- The expectation value of $\mathcal{O}_{11}$ given one eigenvalue x can be written as

$$\mathcal{O}(x) = \mathrm{const} \cdot |x - \overline{x}|^2 e^{-2|x|^2/\sigma^2} \mathrm{Pf}\,[d_{ij}]_{i,j=1}^{2N-2}, \tag{8}$$

  where the d_{ij} are polynomials of degree four in x and $\overline{x}$ if $|i - j| \le 3$, and zero otherwise. $\mathcal{O}(x)$ is represented graphically in Fig. 1.
- Similarly, the expectation of $\mathcal{O}_{12}$ given eigenvalues x_1, x_2 involves the Pfaffian of $(h_{ij})_{i,j=1}^{2N-4}$ with polynomials h_{ij} of degree eight in x_1, $\overline{x_1}$, x_2 and $\overline{x_2}$ if $|i - j| \le 5$, and zero otherwise.

Limiting behaviour (heuristic)

- We expand the expectation values for large N and use convergence of the spectral density ϱ towards the circular law.[6] For $N \to \infty$ we motivate heuristically

$$\mathcal{O}(x) \sim N\varrho_1(x)(1 - |x|^2), \tag{9}$$

$$\mathcal{O}(x_1, x_2) \sim -\varrho_1(x_1)\varrho_1(x_2)\frac{1 - x_1\overline{x_2}}{2N|x_1 - x_2|^4}, \tag{10}$$

  with ϱ_1 the uniform measure on the unit circle, i.e. the limiting spectral density in the bulk.
- For x approaching the origin, we obtain that $\mathcal{O}(x)/\varrho(x)$ grows linearly in N.

Conclusions

- For finite N, we produce explicit formulae for the conditional expectations $\mathcal{O}(x)$ and $\mathcal{O}(x_1, x_2)$, which closely resemble Chalker's and Mehlig's results.
- We derive the limiting distributions of $\mathcal{O}(x)$ and $\mathcal{O}(x_1, x_2)$, which are **identical** to Chalker's and Mehlig's results.
- We show linear growth of $\mathcal{O}(x)/\varrho(x)$ with $x \to 0$ for $N \to \infty$.

Outlook

- Our work uses heavily the iid Gaussian measure of the matrix entries. More general results in the context of Girko's elliptic ensemble and induced Ginibre ensembles are conceivable.
- For $x \in \mathrm{i}\mathbb{R}$, one can resolve $\mathcal{O}(x)$ for small $|x|$ and large N by expanding Eq. (8).
- Rigorous treatment of the asymptotic behaviour of $\mathcal{O}(x)$ and $\mathcal{O}(x_1, x_2)$ has not been covered by our work, but it is addressed by Dubach.[4]
- Quaternionic non-Hermitian operators are applied in QFT and QCD in more general forms than Ginibre's ensembles.[7] One example is two-colour QCD displaying *weak* and *strong* non-Hermiticity.[8]

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