Non-intersecting Brownian motions in presence of a moving boundary

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In collaboration with [1]

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Introduction

The survival of a Brownian path under a moving boundary is a rich problem with many applications. The particular case where the boundary expands critically as $W\sqrt{t}$ exhibits interesting behaviour, with non-universal power-law decay of the survival probability [2, 3, 4].

In this poster, we present a computation of the decay exponent in the case of N non-intersecting Brownians.

An $N = 1$ statistics application

Quantifying the probability for a Brownian particle to survive under a square-root absorbing barrier has an interesting application to the refinement of the Kolmogorov-Smirnov distribution test, as showed by Bouchaud and Chicheportiche [5]. This result improves the sensitivity in the tails, where the original test does not perform well.

Conclusion

The decay exponent for N non-crossing Brownians under a square-root barrier can be expressed as the ground-state energy of a many-body hamiltonian. Small W and large N limits can then be computed through perturbative or semi-classical computations. These results also shed light on the dynamics of random matrices through the connection with Dyson's Brownian motion.

Further N-particle problems include the decay of the survival probability in presence of an expanding cage, or with a linearly expanding barrier.

References

[1] T. Gautié, P. Le Doussal, S. N. Majumdar, and G. Schehr, "To be published,"

[2] S. Redner and P. L. Krapivsky, "Life and death in an expanding cage and at the edge of a receding cliff," *Amer. J Phys.*, vol. 64, 1996.

[3] S. Redner, *A Guide to First-Passage Processes.* Cambridge University Press, 2001.

[4] K. Uchiyama, "Brownian first exit from and sojourn over one sided moving boundary and application," *Z. Wahrscheinlichkeit.*, vol. 54, no. 1, 1980.

[5] R. Chicheportiche and J. P. Bouchaud, "Weighted Kolmogorov-Smirnov test: Accounting for the tails," *PRE*, vol. 86, no. 4, 2012.

[6] A. J. Bray and K. Winkler, "Vicious walkers in a potential," *J. Phys. A: Gen. Phys.*, vol. 37, 2004.

[7] P. Le Doussal, S. N. Majumdar, and G. Schehr, "Periodic Airy process and equilibrium dynamics of edge fermions in a trap," *Ann. Phys.*, vol. 383, 2017.

[8] J. Rambeau and G. Schehr, "Distribution of the time at which N vicious walkers reach their maximal height," *PRE*, vol. 83, 2011.

[9] A. J. Bray, S. N. Majumdar, and G. Schehr, "Persistence and First-Passage Properties in Non-equilibrium Systems," *Advances in Physics*, 2013.

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Single particle results [2, 3, 4]

Decay of the survival probability $S(t)$ of a single Brownian path under an absorbing barrier Wt^α :

- Slow barrier $\alpha < \frac{1}{2}$: $S(t) \propto t^{-\frac{1}{2}}$
- Fast barrier $\alpha > \frac{1}{2}$: $S(t) \to \ell > 0$
- Critical case $\alpha = \frac{1}{2}$: Power-law decay with non-universal exponent $\beta(W)$

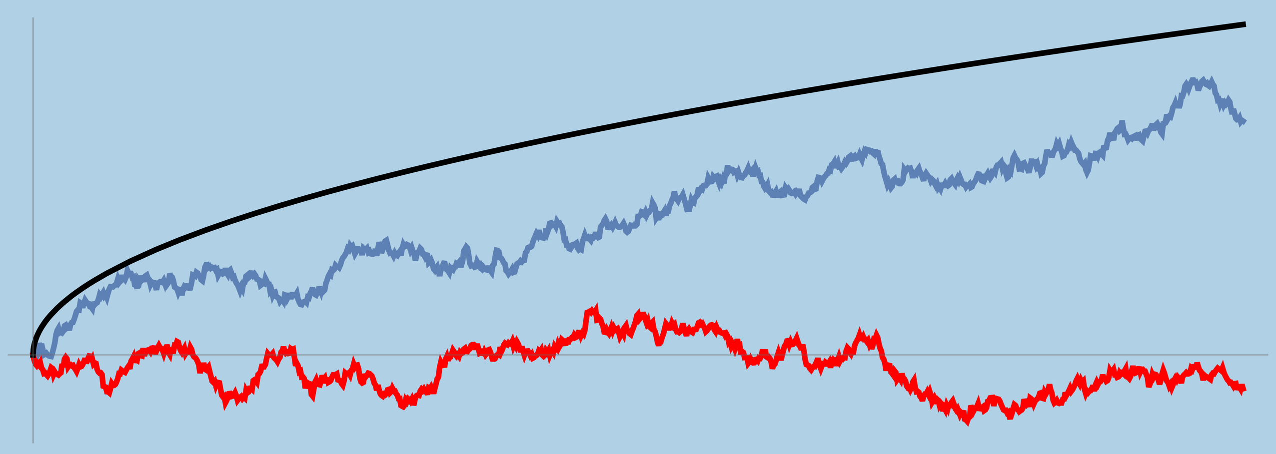
N-particle generalization in the critical case

Model : N vicious Brownian paths under a moving absorbing boundary $x_b(t) = W\sqrt{t}$. What is the persistence/decay exponent $\beta(N, W)$ for the survival probability ?

$$S(t) = \Pr\left(x_1(\tau) < \cdots < x_N(\tau) < W\sqrt{\tau}\ ;\ \forall \tau \in [t_0, t]\right)$$

$$S(t) \propto t^{-\beta(N,W)}$$

Figure 1: Realization for $N = 2$



Results : The exponent can be expressed as the ground-state energy of a system of N fermions in a harmonic trap with an infinite wall at position W. With D_ν the parabolic cylinder function and $E_k(W)$ the k-th solution of $D_{2E_k(W)}(W) = 0$, this is (recovering [6] for $W = 0, \infty$):

$$\beta(N, W) = \sum_{k=0}^{N-1} E_k(W) = \begin{cases} \frac{N(N-1)}{4} & \text{for } W = \infty \\ \frac{N^2}{2} & \text{for } W = 0 \end{cases} \tag{1}$$

$$\hat{H} = \sum_{j=1}^{N} \left(-\frac{1}{2}\frac{\partial^2}{\partial X_j^2} + \frac{1}{8}X_j^2 - \frac{1}{4} \right) + \hat{H}_{\text{wall at } W} \tag{2}$$

$W \to 0$ limit : Perturbation theory gives at second order in W (agreeing with [2] for $N = 1$), where the ΔE_n^2 term can be expressed as a series :

$$\beta(N, W) = \frac{N^2}{2} - W \frac{\sqrt{2}}{3\sqrt{\pi}} \frac{1}{2^{2N}} \frac{N(2N+1)!}{(N!)^2} + W^2 \sum_{n=0}^{N-1} \Delta E_n^2 + \frac{W^2}{8} \tag{3}$$

Large N limit : For large k, the energies E_k can be approximated semi-classically to form a continuum with a scaling function f, such that $E_k = k\ f\left(\frac{W}{\sqrt{k}}\right)$, which gives the exponent through :

$$\beta(N, W) \sim \int_{W/\sqrt{N}}^{W} \frac{2W^4 f(z)}{z^5} dz \qquad \text{with} \quad \pi(1 - f(z)) = \frac{z}{\sqrt{2}}\sqrt{f(z) - \frac{z^2}{8}} + 2f(z)\arcsin\left(\frac{z}{\sqrt{8f(z)}}\right)$$

Application to Random Matrices : The decay exponent for the survival probability of Dyson's Brownian motion in the space of Hermitian ($\beta = 2$) matrices, under a $W\sqrt{t}$ barrier is :

$$\beta_{\text{Dyson}}(N, W) = \beta(N, W) - \frac{N(N-1)}{4} \xrightarrow[W \to \infty]{} 0 \tag{4}$$

Computation details

- Lamperti mapping : $\frac{dx}{dt} = \eta(t) \xrightarrow[t = e^T]{X = \frac{x}{\sqrt{t}}} \frac{dX(T)}{dT} = -\frac{1}{2}X(T) + \xi(T)$
- Quantum mapping of the Ornstein-Uhlenbeck process (at large times *wrt* the gap), where ϕ_0 is the ground-state wavefunction of $\hat{H}$:

$$\left.\begin{array}{l} S_{Br}(t \mid \vec{x_0}, t_0; W\sqrt{t}) \\ S_{OU}(t \mid \vec{X}_0, T_0; W) \end{array}\right\} \sim \left(\frac{t_0}{t}\right)^{\beta(N,W)} N! \int_{X_{\max} < W} d^N\vec{X}\ e^{-\frac{1}{4}\sum_j (X_j^2 - {X_0}_j^2)} \phi_0(\vec{X})\phi_0^*(\vec{X}_0)$$

- For $W \sim 0$, perturbation theory gives :

$$\beta(N, W) = \beta(N, 0) + W \sum_{n=0}^{N-1} \Delta E_n^1 + W^2 \sum_{n=0}^{N-1} \Delta E_n^2 + \frac{W^2}{8} \text{ with } \begin{cases} \Delta E_n^1 = \langle n | \Delta\hat{H} | n \rangle \\ \Delta E_n^2 = \sum_{k \neq n} \frac{\left|\langle k | \Delta\hat{H} | n \rangle\right|^2}{E_n - E_k} \end{cases}$$

- Dyson's Brownian motion (for $\beta = 2$), and its connection to the non-crossing Brownians [8] :

$$dx_i = dB_i + \sum_{1 \le j \le n : j \neq i} \frac{dt}{x_i - x_j} \qquad \mathcal{P}_{\text{Dyson}}(\vec{x}, t | \vec{y}, t_0) = \frac{\prod_{i<j}(x_j - x_i)}{\prod_{i<j}(y_j - y_i)} \mathcal{P}_{\text{Non-crossing}}(\vec{x}, t | \vec{y}, t_0)$$