# Fluctuations of observables for free fermions in a harmonic trap at finite temperature

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## A simple model: 1D harmonic trap

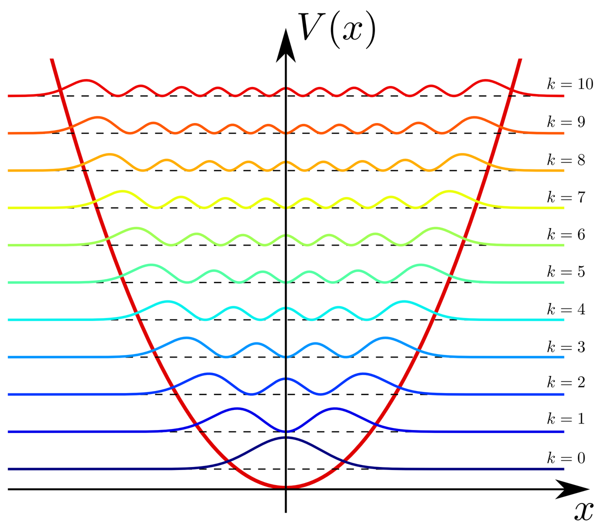
$$\hat{\mathcal{H}} = \sum_{i=1}^{N} \hat{\mathcal{H}}_i \,, \; \hat{\mathcal{H}}_i = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2$$

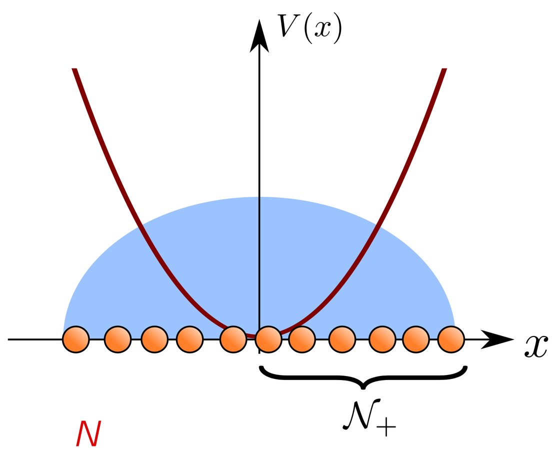
One-particle wave functions:

$$\mathcal{H}_i \psi_k = \varepsilon_k \psi_k$$

$$\Rightarrow \begin{cases} \varepsilon_k = k + \frac{1}{2} \\ \psi_k(x) = c_k\, H_k(x)\, \mathrm{e}^{-x^2/2} \end{cases}$$

$H_n \;\to$ Hermite polynomials




$$\mathcal{N}_+ = \sum_{i=1}^{N} f(x_i) \,, \quad f(x) = \Theta(x)$$

$\to$ linear statistics

**Q:** $\mathrm{Var}(\mathcal{N}_+)$ ?

## Many-body wave functions

Antisymmetric combinations of the $\{\psi_i\}$
$\to$ Slater determinants

Labelled by occupation numbers $\{n_i = 0, 1\}_{i\in\mathbb{N}}$

$$\Psi_{\{n_i\}}(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det[\psi_{k_i}(x_j)]$$

$\{k_i\}$ list of occupied levels ($n_{k_i} = 1$)

Energy $E_{\{n_i\}} = \sum_i n_i \varepsilon_i$

## Zero temperature: Random Matrix Theory

Ground state: ($n_0 = \dots = n_{N-1} = 1$, $n_{i\geq N} = 0$)

$$\Psi_0(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det[\psi_{i-1}(x_j)]_{1\leq i,j\leq N}$$

$$\propto \prod_{i<j}(x_i - x_j) \prod_{k=1}^{N} \mathrm{e}^{-x_k^2/2}$$

$\Rightarrow$ Prob. density:

$$|\Psi_0(\{x_i\})|^2 \propto \prod_{i<j}(x_i - x_j)^2 \prod_{k=1}^{N} \mathrm{e}^{-x_k^2}$$

positions of fermions in a harmonic trap at $T = 0$
$\Updownarrow$
eigenvalues of GUE

$\mathcal{N}_+ \leftrightarrow$ index of GUE random matrices

Result for the variance for large $N$ (see Ref. [1])

$$\mathrm{Var}(\mathcal{N}_+)|_{T=0} \simeq \frac{1}{2\pi^2} \ln N + \dots$$

## $T > 0$: statistical physics

$T > 0$: all $|\Psi_{\{n_i\}}\rangle$ contribute with weight $\mathcal{P}(\{n_i\})$

- Grand canonical ensemble: $T$, $\mu$ fixed

$$\mathcal{P}(\{n_i\}) \propto \mathrm{e}^{-\beta\sum_i n_i(\varepsilon_i - \mu)} \quad \to n_i \text{ indep.}$$

- Canonical ensemble: $T$, $N$ fixed

$$\mathcal{P}(\{n_i\}) \propto \mathrm{e}^{-\beta\sum_i n_i\varepsilon_i}\, \delta_{\sum_i n_i, N} \quad \to n_i \text{ correl.}$$

$$\overline{n_i n_j} = \frac{\mathrm{e}^{\beta\varepsilon_i}\overline{n_i} - \mathrm{e}^{\beta\varepsilon_j}\overline{n_j}}{\mathrm{e}^{\beta\varepsilon_i} - \mathrm{e}^{\beta\varepsilon_j}} \qquad [3, 4, 5]$$

## Quantum + thermal fluctuations

① Fluctuations of $\mathcal{N}_+$ in the state $|\Psi_{\{n_i\}}\rangle$
$\to$ Quantum fluctuations

② $|\Psi_{\{n_i\}}\rangle$ picked from the distribution $\mathcal{P}(\{n_i\})$
$\to$ Thermal fluctuations

$\Rightarrow$ Usual quantum and thermal average

$$\mathrm{Var}_{\mathrm{c,g}}(\mathcal{N}_+) = \overline{\langle(\mathcal{N}_+)^2\rangle_{\Psi_{\{n_i\}}}}^{\mathrm{c,g}} - \left(\overline{\langle\mathcal{N}_+\rangle_{\Psi_{\{n_i\}}}}^{\mathrm{c,g}}\right)^2$$

## Quantum fluctuations: Determinantal point processes

$$|\Psi_{\{n_i\}}(x_1, \dots, x_N)|^2 = \frac{1}{N!} \det\left[K(x_i, x_j; \{n_i\})\right]_{1\leq i,j\leq N} \,, \quad K(x, y; \{n_i\}) = \sum_i n_i \psi_i^*(x)\psi_i(y)$$

$$\int K(x, y; \{n_i\}) K(y, z; \{n_i\}) \mathrm{d}y = K(x, z; \{n_i\}) \quad \text{since} \quad n_i^2 = n_i$$

positions of fermions in a given state $|\Psi_{\{n_i\}}\rangle$ form a determinantal point process [2]

$$\langle\mathcal{N}_+\rangle_{\Psi_{\{n_i\}}} = \sum_i n_i A_{ii} \,, \qquad \langle(\mathcal{N}_+)^2\rangle_{\Psi_{\{n_i\}}} = \sum_i n_i B_i + \sum_{i,j} n_i n_j (A_{ii}A_{jj} - (A_{ij})^2)$$

$$A_{ij} = \int \psi_i^*(x)\psi_j(x) f(x)\mathrm{d}x \,, \quad B_i = \int \psi_i^*(x)\psi_i(x) f(x)^2 \mathrm{d}x$$

## General formula for the variance

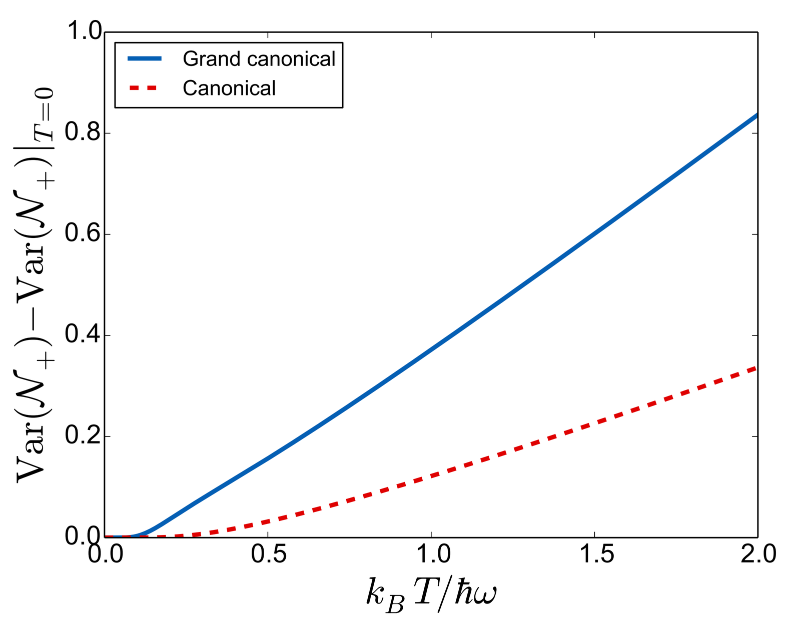
$$\mathcal{L} = \sum_i f(x_i)$$

$$\mathrm{Var}_{\mathrm{c,g}}(\mathcal{L}) = \sum_i \overline{n_i}^{\mathrm{c,g}} B_i - \sum_{i,j} \overline{n_i}^{\mathrm{c,g}}\,\overline{n_j}^{\mathrm{c,g}} (A_{ij})^2 + \sum_{i\neq j} \mathrm{Cov}_{\mathrm{c,g}}(n_i, n_j)(A_{ii}A_{jj} - (A_{ij})^2)$$

## Quantum regime

$$\mathrm{Var}(\mathcal{N}_+) \simeq \mathrm{Var}(\mathcal{N}_+)|_{T=0} + F_Q\left(\frac{T_Q}{T}\right) + \begin{cases} 0 & \text{(c)} \\ \frac{1}{4}\mathrm{Var_g}(N) & \text{(g)} \end{cases}$$

$$F_Q\left(\xi = \frac{T_Q}{T}\right) = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{2n-1} \frac{1}{\mathrm{e}^{(2n-1)\xi} - 1}$$

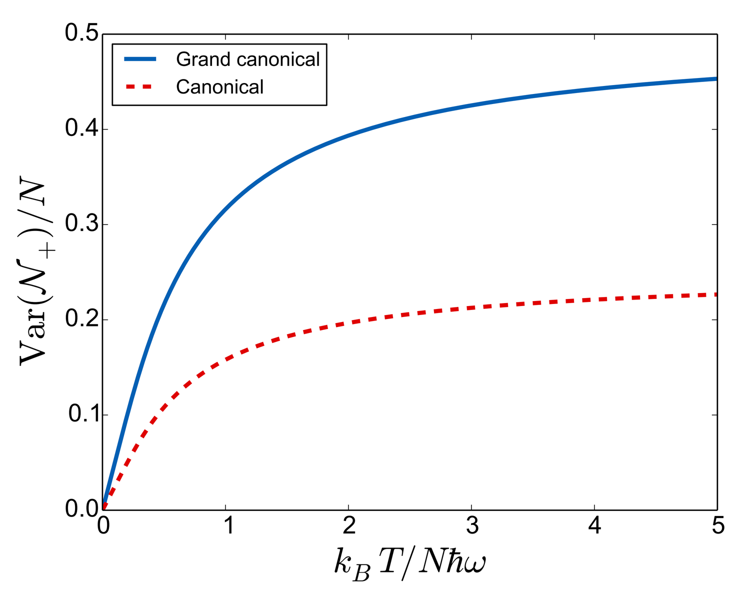



**Universal**

## Thermal regime

$$\mathrm{Var}(\mathcal{N}_+) \simeq \overline{N}^{\mathrm{c,g}} F_T\left(\frac{T_F}{T}\right) + \begin{cases} 0 & \text{(c)} \\ \frac{1}{4}\mathrm{Var_g}(N) & \text{(g)} \end{cases}$$

$$F_T\left(y = \frac{T_F}{T}\right) = \frac{1 - \mathrm{e}^{-y}}{4y} = \frac{1}{4\overline{N}^{\mathrm{g}}} \mathrm{Var_g}(N)$$




**Not Universal**

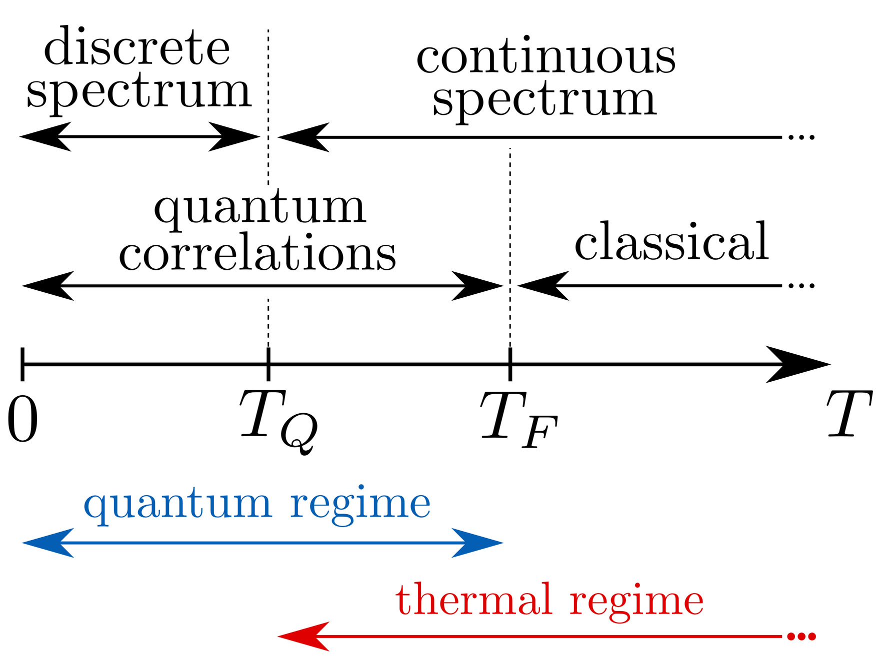
## Two temperature scales

- $T_Q = 1$ (spacing of energies $\varepsilon_k$)

Quantum regime: $T/T_Q$ fixed and $N \to \infty$

- $T_F = N$ (Fermi temperature)

Thermal regime: $T/T_F$ fixed and $N \to \infty$




## References

[1] S. N. Majumdar, C. Nadal, A. Scardicchio, P. Vivo, Phys. Rev. Lett. **103**, 220603 (2009)
[2] D. S. Dean, P. Le Doussal, S. N. Majumdar, G. Schehr, Phys. Rev. Lett. **114**, 110402 (2015)
[3] K. Schönhammer, Phys. Rev. A **96** , 012102 (2017)
[4] O. Giraud, A. Grabsch, C. Texier, Phys. Rev. A **97**, 053615 (2018)
[5] A. Grabsch, S. N. Majumdar, G. Schehr, C. Texier, SciPost Phys. **4**, 014 (2018)