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Laguerre polynomials and transitional asymptotics of the modified Korteweg-de Vries equation for step-like initial data

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Cauchy problem for the modified Korteweg – de Vries equation

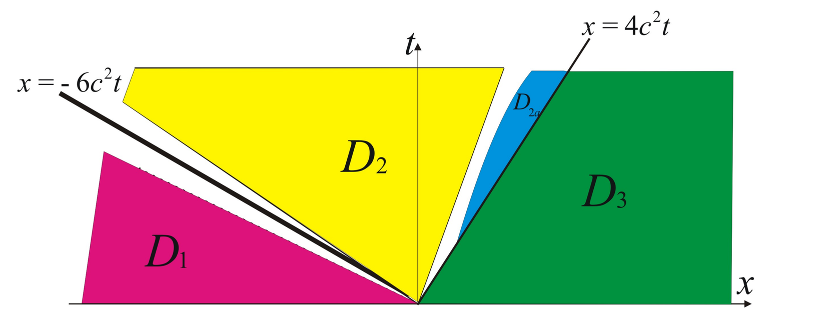
We study the asymptotic behavior for large t of the Cauchy problem solution for MKdV equation

$$q_t(x,t)+6q^2(x,t)q_x(x,t)+q_{xxx}(x,t)=0,\ x\in\mathbb{R},\ t\geq 0, \tag{1}$$

$$q(x,0)=q_0(x),\quad q_0(x)\to c>0, x\to-\infty,\quad q_0(x)\to 0, x\to+\infty. \tag{2}$$

We assume that

1. exponential convergence to the background constants at the infinity
$$\int_{-\infty}^{0}\mathrm{e}^{-Cx}|q_0(x)-c|\mathrm{x}+\int_{0}^{+\infty}\mathrm{e}^{-Cx}|q_0(x)|\mathrm{x}<\infty,$$
2. generic behavior of the associated spectral function $a(k)$ at the edge point $k=\mathrm{i}c$,
3. no discrete spectrum,



Division of the $x,t>0$-half-plane into regions with qualitatively different asymptotics.

These assumptions are satisfied for example for the pure-step function $q_0(x)=\begin{cases}c>0, & x<0,\\ 0, & x>0.\end{cases}$ (3)

Results

The asymptotic behavior of the solution of the Cauchy problem (1)-(2) has qualitatively different forms in different domains of the x,t plane.

D_1 : For $x<-(6c^2+\varepsilon)t$ the solution of the mKdV equation has the following asymptotic behavior

$$q(x,t)=c+\frac{1}{\sqrt{t}}\cdot\sqrt{\frac{\nu(\xi)\sqrt{\frac{c^2}{2}-\xi}}{3\left(-\xi-\frac{c^2}{2}\right)}}\cos\left(16t\left(\frac{c^2}{2}-\xi\right)^{3/2}-\nu(\xi)\log\left(\frac{192t\left(-\xi-\frac{c^2}{2}\right)^2}{\sqrt{\frac{c^2}{2}-\xi}}\right)+\varphi(\xi)\right)+\mathrm{O}\left(t^{-1}\right),\ t\to+\infty.$$

Here

$$\xi=\frac{x}{12t},\quad \nu(\xi)=\frac{1}{2\pi}\log\left(1+r\left(\sqrt{-\xi-\frac{c^2}{2}}\right)\right),\quad \varphi(\xi)\text{ is a bounded function,}$$

and $r(k)$ is the reflection coefficient associated with the initial data $q_0(x)$ via the spectral problem (5a).

D_2 :

For $-(6c^2-\varepsilon)t<x<(4c^2-\varepsilon)t$: $\quad q(x,t)=q_{el}(x,t)+\mathcal{O}(t^{-1})$, $t\to+\infty$,

For $-(6c^2-\varepsilon)t<x<4c^2t-\beta t^\sigma\ln t,\quad 0<\sigma<1$: $\quad q(x,t)=q_{el}(x,t)+\mathcal{O}(t^{-\sigma})$, $t\to+\infty$,

where the main term of asymptotics of the solution of the mKdV equation is given by modulated elliptic wave

$$q_{el}(x,t)=\sqrt{c^2-d^2(\xi)}\frac{\Theta(\pi\mathrm{i}+\mathrm{i}tB(\xi)+\mathrm{i}\Delta(\xi)|\tau(\xi))}{\Theta(\mathrm{i}tB(t,\xi)+\mathrm{i}\Delta(\xi)|\tau(\xi))}. \tag{4}$$

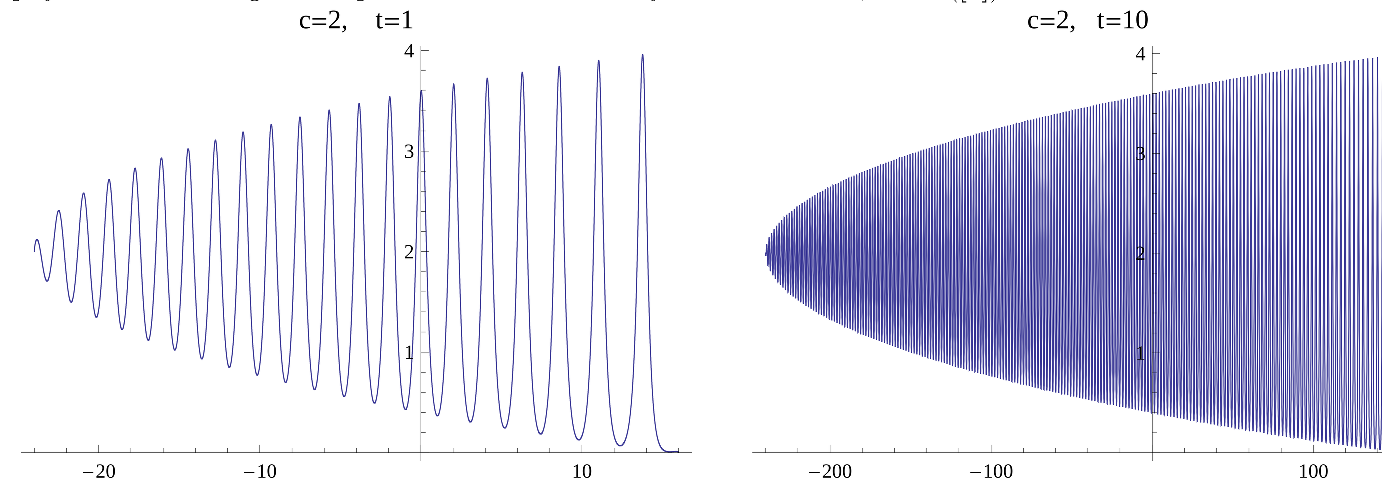
Here $\Theta(z|\tau)$ is the standard theta-function, $d=d(\xi),\tau=\tau(\xi)$ are increasing functions, defined for $\xi\in\left(\frac{-c^2}{2},\frac{c^2}{3}\right)$.
$d\left(\frac{-c^2}{2}\right)=0$, $\tau\left(\frac{-c^2}{2}\right)=-\infty$, $d\left(\frac{c^2}{3}\right)=c$, $\tau\left(\frac{c^2}{3}\right)=-0$, $B(\xi)$ is a complete elliptic integral,

$$\Delta(\xi)=\mathrm{i}\int_{\mathrm{i}c}^{\mathrm{i}d}\frac{\log a_+(s)a_-(s)\mathrm{d}s}{\mathrm{w}_+(s,\xi)}\left(\int_{\mathrm{i}d}^{0}\frac{\mathrm{d}s}{\mathrm{w}(s,\xi)}\right)^{-1},$$

where $a^{-1}(k)$ is the transmission coefficient associated with $q_0(x)$ via (5a).

D_3 : For $x\geq 4c^2t$ the asymptotic behavior of the solution is $q(x,t)=\mathrm{O}\left(\frac{\mathrm{e}^{-8ct(3\xi-c^2)}}{\sqrt{t}}\right)$, $t\to+\infty$.

⋆ On the physical level of rigor this problem was studied by R. F. Bikbaev, 1995 ([1]).



q_{el} for the initial function (3).

D_{2a} : Let $\beta\in[\frac{1}{K},K]$, $K>0$, and let (x,t) lie on the curve $x=4c^2t-\beta t^\sigma\ln t$. Define $n=n(t)\sim[\frac{\beta t^\sigma}{1-\sigma}]$. Then, as $t\to\infty$,

$$0\leq\sigma\leq\sigma_0<\frac{1}{2}:\quad q(x,t)=\frac{2c}{\cosh\left(2c(x-4c^2t)+(2n+\frac{3}{2})\ln t+\alpha_n\right)}+\mathcal{O}\left(t^{2\sigma-1}\ln^2 t\right),$$

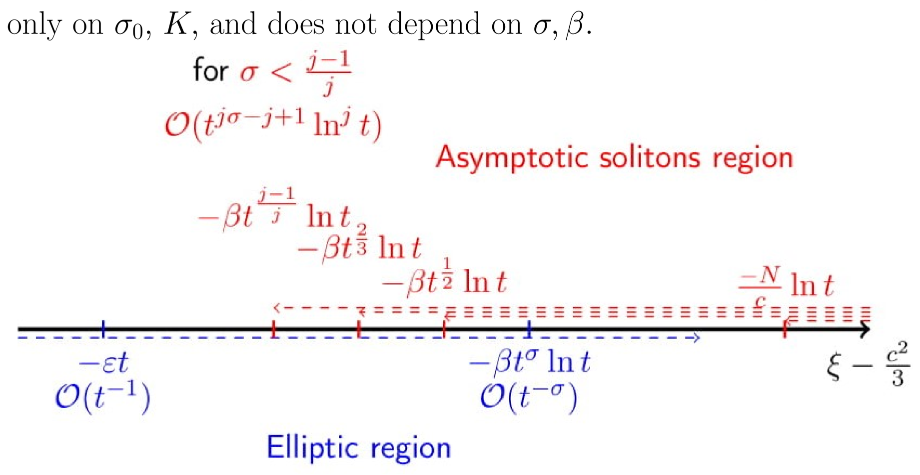
$$0\leq\sigma\leq\sigma_0<\frac{2}{3}:\quad q(x,t)=\frac{2c}{\cosh\left(2c(x-4c^2t)+(2n+\frac{3}{2})\ln t+\alpha_n+\alpha_n^{(2/3)}\right)}+\mathcal{O}\left(t^{3\sigma-2}\ln^3 t+t^{\sigma-1}\right),$$

$$0\leq\sigma\leq\sigma_0<\frac{3}{4}:\quad q(x,t)=\frac{2c}{\cosh\left(2c(x-4c^2t)+(2n+\frac{3}{2})\ln t+\alpha_n+\alpha_n^{(2/3)}+\alpha_n^{(3/4)}\right)}+\mathcal{O}\left(t^{4\sigma-3}\ln^4 t+t^{\sigma-1}\right).$$

$$\alpha_n^{(2/3)}=\frac{n^2}{4c^3t}\ln\frac{32c^3t}{\mathrm{e}^2n},\quad \alpha_n^{(3/4)}=\frac{n^3}{64c^6t^2}\left(-31+\frac{25}{2}\ln\frac{32c^3\mathrm{e}t}{n}-\left(\ln\frac{32c^3\mathrm{e}t}{n}\right)^2\right),$$

where the constant in $\mathcal{O}$ estimate depends only on σ_0, K, and does not depend on σ,β.

⋆ For $\sigma=0$ the asymptotics in D_{2a} was studied by E. Ya. Khruslov and V. P. Kotlyarov ([3]). Such a logarithmic zone was first studied for the KdV equation by E. Ya. Khruslov ([4]).



Zone scales in transition region.

Main method: the Riemann – Hilbert (RH) problem

mKdV equation (1) is a compatibility condition for the 2×2 matrix system (M. Wadati, 1972)

$$\mathrm{i}\sigma_3\frac{\mathrm{d}}{\mathrm{d}x}\Psi(x,t,k)-\begin{pmatrix}0 & \mathrm{i}q(x,t)\\ \mathrm{i}q(x,t) & 0\end{pmatrix}\Psi(x,t,k)=k\Psi(x,t,k),\ \sigma_3=\begin{pmatrix}1&0\\0&1\end{pmatrix}, \tag{5a}$$

$$\mathrm{i}\sigma_3\frac{\mathrm{d}}{\mathrm{d}t}\Psi(x,t,k)-\mathrm{i}\sigma_3\hat{Q}(x,t,k)\Psi(x,t,k)=4k^3\Psi(x,t,k). \tag{5b}$$

Denote by E_0,E_c the solutions of system (5) for backgrounds $q=c$ and $q=0$. Function E_c is analytic in $k\in\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$.
Matrix Jost solutions of (5) has the following asymptotic behavior

$$\Psi_0(x,t,k)\sim E_0(x,t,k),\quad x\to+\infty,\quad \mathrm{Im}k=0,\qquad \Psi_c(x,t,k)\sim E_c(x,t,k),\quad x\to+\infty,\quad \mathrm{Im}\sqrt{k^2+c^2}=0.$$

Define a sectionally analytical matrix $M(x,t,k)$

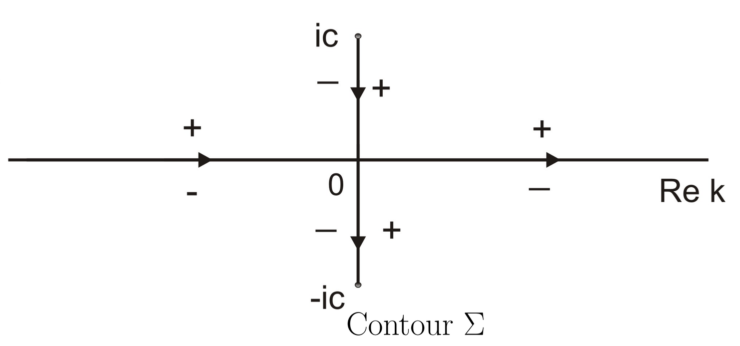
$$M(x,t,k)=\begin{cases}\left(a^{-1}(k)\Psi_{c,1}(x,t,k)\mathrm{e}^{\mathrm{i}kx+4\mathrm{i}k^3t},\ \Psi_{0,2}(x,t,k)\mathrm{e}^{-\mathrm{i}kx-4\mathrm{i}k^3t}\right), & \mathrm{Im}\sqrt{k^2+c^2}>0\\ \left(\Psi_{0,1}(x,t,k)\mathrm{e}^{\mathrm{i}kx+4\mathrm{i}k^3t},\ \overline{a^{-1}(\overline{k})}\Psi_{c,2}(x,t,k)\mathrm{e}^{-\mathrm{i}kx-4\mathrm{i}k^3t}\right), & \mathrm{Im}\sqrt{k^2+c^2}<0,\end{cases}$$

where $\Psi_{c,j},\Psi_{0,j}$, $j=1,2$, are the columns of matrices Ψ_c, Ψ_0. Then $M(x,t,k)$ solves the following Riemann – Hilbert problem on the contour Σ (x,t – parameters):

1. $M(x,t,k)$ is analytic $\mathbb{C}\setminus\Sigma$, continuous up to the boundary, and $M(x,t,k)=I+O(k^{-1})$, $\quad k\to\infty$.
2. $M_-(x,t,k)=M_+(x,t,k)J(x,t,k)$, $\quad k\in\Sigma\setminus(\{0\}\cup\{\mathrm{i}c\}\cup\{-\mathrm{i}c\})$, where

$$J(x,t,k)=\begin{pmatrix}1 & r(k)e^{-2\mathrm{i}kx-8\mathrm{i}k^3t}\\ -r(k)e^{2\mathrm{i}kx+8\mathrm{i}k^3t} & 1+|r(k)|^2\end{pmatrix},\quad k\in\mathbb{R},$$
$$=\begin{pmatrix}1 & 0\\ f(k)e^{2\mathrm{i}kx+8\mathrm{i}k^3t} & 1\end{pmatrix},\quad k\in(0,\mathrm{i}c),$$
$$=\begin{pmatrix}1 & f(k)e^{-2\mathrm{i}kx-8\mathrm{i}k^3t}\\ 0 & 1\end{pmatrix},\quad k\in(0,-\mathrm{i}c),$$

$f(k)=r_-(k)-r_+(k)$



Contour Σ

The solution of the mKdV equation can be found via the solution of this RH problem by formula

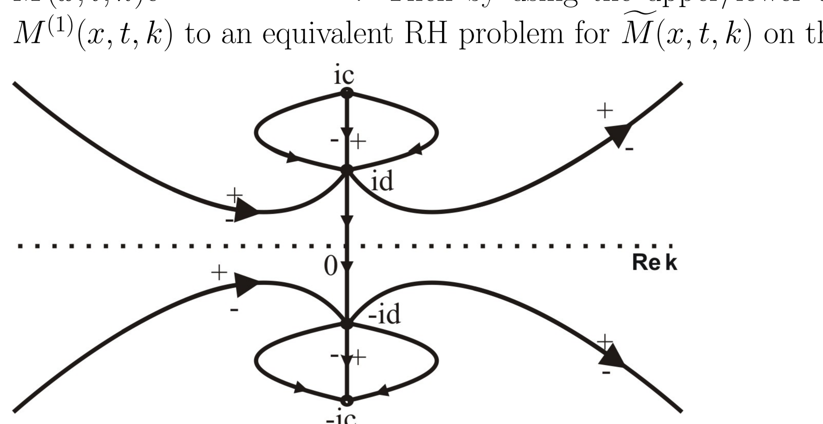
$$q(x,t)=\lim_{k\to\infty}2\mathrm{i}kM_{21}(x,t,k).$$

Asymptotic analysis of the solution of the RH problem

For asymptotic analysis we distinguish the domains D_1, D_2, D_3 in x,t half plane: Particularly, in D_2 we use an auxiliary function $g(k,\xi)$, that has the following properties:

- $g(k,\xi)$ is an odd analytic function in $k\in\mathbb{C}\setminus[\mathrm{i}c,-\mathrm{i}c]$,
- $g_-(k,\xi)+g_+(k,\xi)=0$, $\quad k\in(\mathrm{i}c,\mathrm{i}d(\xi))\cup(-\mathrm{i}d(\xi),-\mathrm{i}c)$,
  $g_-(k,\xi)-g_+(k,\xi)=B(\xi)$, $\quad k\in(\mathrm{i}d(\xi),-\mathrm{i}d(\xi))$,
- $g(k,\xi)-4k^3-12\xi k=\mathrm{O}(k^{-1})$, $k\to\infty$.

$g(k,\xi)$ can be presented as a normalized Abelian integral of the second kind on the Riemann surface of function $\mathrm{w}(k,\xi)$ with the b-period $B(\xi)$.
With the help of this function we make the first transformation of the solution of given RH problem $M^{(1)}(x,t,k)=M(x,t,k)\mathrm{e}^{\mathrm{i}t(g(k,\xi)-4\mathrm{i}k^3-\mathrm{i}kx)}$. Then by using the upper/lower triangular factorizations as in [2], we reduce the RH problem for $M^{(1)}(x,t,k)$ to an equivalent RH problem for $\widetilde{M}(x,t,k)$ on the contour $\widetilde{\Sigma}$:



$$\widetilde{M}_-(x,t,k)=\widetilde{M}_+(x,t,k)\widetilde{J}(x,t,k),$$

$$\widetilde{J}=\begin{cases}\begin{pmatrix}0&\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}, & k\in(\mathrm{i}c,\mathrm{i}d)\cup(-\mathrm{i}d,-\mathrm{i}c),\\ \mathrm{e}^{(\mathrm{i}tB(\xi)+\mathrm{i}\Delta(\xi))\sigma_3}, & k\in(\mathrm{i}d,-\mathrm{i}d),\\ I+\bar{\mathrm{o}}(1), & k\in\widetilde{\Sigma}\setminus[\mathrm{i}c,-\mathrm{i}c],\ t\to\infty.\end{cases}$$

Contour $\widetilde{\Sigma}$

As $t\to\infty$, the jump matrix $\widetilde{J}(x,t,k)$ becomes identity matrix on $\widetilde{\Sigma}\setminus[\mathrm{i}c,-\mathrm{i}c]$, and we come to the model problem on the contour $[\mathrm{i}c,-\mathrm{i}c]$, which can be solved explicitly. The solution of this problem gives us the main term (4) of asymptotics of the solution of problem (1)-(2).

Analysis in transition zone

For (x,t) lying on the logarithmic curve $x=4c^2t-\rho\ln t$, we proceed with the same factorization as for the region $\xi>\frac{c^2}{3}$. The main contribution comes from the intervals $\pm(\mathrm{i}c,\mathrm{i}(c-r))$, where $r>0$ is small. Making a scaling change of variable $\zeta=t\cdot\mathrm{i}(k-\mathrm{i}c)$ in a vicinity of the point $k=\mathrm{i}c$, we find that the jump matrix on the interval $k\in(\mathrm{i}c,\mathrm{i}(c-r))$ equals

$$(\varphi(k)t^\gamma)^\sigma\begin{pmatrix}1 & 0\\ \sqrt{\zeta}\mathrm{e}^{-\zeta} & \end{pmatrix}(\varphi(k)t^\gamma)^{-\sigma},\zeta\in(0,+\infty),$$

where φ is a regular function, and $\gamma=c\rho-\frac{1}{4}$.A solution to such jump problem can be constructed out of the monic Laguerre polynomials of the order $\frac{1}{2}$ $\pi_n(\zeta)$ and the ir Cauchy transforms $C_n(\zeta)$

$$L_n(\zeta)=\begin{pmatrix}\beta_nC_{n-1}(\zeta) & \beta_n\pi_{n-1}(\zeta)\\ C_n(\zeta) & \pi_n(\zeta)\end{pmatrix},\quad C_n(\zeta)=\frac{1}{2\pi\mathrm{i}}\int_0^{+\infty}\frac{\pi_n(s)\sqrt{s}\mathrm{e}^{-s}\mathrm{d}s}{s-\zeta}.$$

For those γ with $\{\gamma\}\in[0,\frac{1}{2}]$, we construct an approximation solution to the RHP as follows. Let $n=\lfloor\gamma\rfloor$ be the integer part of γ. Define

$$M_\infty^{(1)}=\begin{cases}G(k)\left(\frac{k-\mathrm{i}c}{k+\mathrm{i}c}\right)^{-n\sigma_3}, & |k\mp\mathrm{i}c|>r,\\ G(k)B(k)\begin{pmatrix}1&0\\ \frac{R_1}{\zeta}&1\end{pmatrix}L_n(\zeta)(\varphi(k))^{\sigma_3}t^{\gamma\sigma_3}, & |k-\mathrm{i}c|<r,\\ G(k)B_d\begin{pmatrix}1&\frac{R_1}{\zeta_d}\\ 0&1\end{pmatrix}L_d(\mathrm{i}\varphi_d)^{-\sigma_3/2}t^{-\gamma\sigma_3}, & |k+\mathrm{i}c|<r,\end{cases}$$

Here $G(k)=I+\frac{A}{k-\mathrm{i}c}+\frac{\widetilde{A}}{k+\mathrm{i}c}$ is a function, which is introduced in order to cancel the singularity at $k=\mathrm{i}c$, caused by the triangular matrix factor. $B(k)$ is a regular function, whose goal is to make the error matrix $M_{err}=MM_\infty^{-1}$ to be close to the identity matrix.
Each n corresponds to a different asymptotic soliton. The soliton's peak is situated at $\{\gamma\}=\frac{1}{2}$.
For those γ with $\{\gamma\}\in(\frac{1}{2},1)$ the construction of an approximate RHP solution is similar, with changing the triangularity of the matrix factors.
Finally, for (x,t) on the line $x=4c^2t-\beta t^\sigma\ln t$, which correspond to $n\sim t^\sigma$, one needs to make a more sophisticated local change of the variable k, in order to avoid the presence of terms $(1+\mathcal{O}(k-\mathrm{i}c))^n$ in the jump matrix for the error matrix M_{err}.

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