A Riemann-Hilbert approach to the Muttalib-Borodin ensemble

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The model

The Muttalib-Borodin ensemble [2] with parameter $\theta > 0$ and weight function w is the following probability density function:

$$\frac{1}{Z_n}\prod_{j<k}(x_k - x_j)(x_k^\theta - x_j^\theta)\prod_{j=1}^{n} w(x_j), \quad x_j \geq 0.$$

We consider an n-dependent weight function

$$w(x) = x^\alpha e^{-nV(x)}$$

with $\alpha > -1$ and an external field V. The ensemble is a **determinantal point process**, it can be written as

$$\det\left(K_{V,n}^{\alpha,\theta}(x_i, x_j)\right)_{1\leq i,j\leq n}$$

where $K_{V,n}^{\alpha,\theta}(x,y)$ is the so-called correlation kernel.

Known result and main interest

Our main interest is to study the large n behavior of $K_{V,n}^{\alpha,\theta}(x,y)$. Borodin [2] computed the **hard edge scaling limit** for the Laguerre case, namely if $V(x) = x$, then

$$\lim_{n\to\infty}\frac{1}{n^{1+1/\theta}}K_{V,n}^{\alpha,\theta}\left(\frac{x}{n^{1+1/\theta}}, \frac{y}{n^{1+1/\theta}}\right) = \mathbb{K}^{(\alpha,\theta)}(x,y)$$

with limiting correlation kernel

$$\mathbb{K}^{(\alpha,\theta)}(x,y) = \theta y^\alpha \int_0^1 J_{\frac{\alpha+1}{\theta},\frac{1}{\theta}}(ux) J_{\alpha+1,\theta}\left((uy)^\theta\right) u^\alpha du$$

where

$$J_{a,b}(x) = \sum_{j=0}^{\infty}\frac{(-x)^j}{j!\Gamma(a+bj)}.$$

The same limit turns up in products of random matrices [1,4,5]. From these models and others [7] we know that the limit can be expressed in terms of **Meijer G-functions**.

By **universality** the limit is expected to hold for a much larger class of external fields V and our goal is to prove this.

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Equilibrium problem

There is a corresponding equilibrium problem: minimize

$$\frac{1}{2}\iint \log\frac{1}{|x-y|}d\mu(x)d\mu(y) + \frac{1}{2}\iint \log\frac{1}{|x^\theta - y^\theta|}d\mu(x)d\mu(y) + \int V(x)d\mu(x)$$

among all probability measures μ on $[0,\infty)$. There exists a unique absolutely continuous solution $\mu_{V,\theta}^*$, called the **equilibrium measure** [6].

We call the external field V one-cut θ-regular if $\mu_{V,\theta}^*$ is supported on one interval $[0,q]$ for some $q > 0$ with a density that is positive on $(0,q)$ and satisfies

$$\frac{d\mu_{V,\theta}^*(s)}{ds} = \begin{cases} c_0 s^{-\frac{1}{\theta+1}}(1+o(1)) & \text{as } s \to 0+, \\ c_1(q-s)^{1/2}(1+o(1)) & \text{as } s \to q-, \end{cases}$$

Main Theorem

Let $\alpha > -1$, $\theta = 1/2$, and let $V : [0,\infty) \to \mathbb{R}$ be a real analytic external field that is one-cut θ-regular (as above). Then for $x, y \in (0,\infty)$ we have

$$\lim_{n\to\infty}\frac{1}{(cn)^3}K_{V,n}^{\alpha,\frac{1}{2}}\left(\frac{x}{(cn)^3}, \frac{y}{(cn)^3}\right) = \mathbb{K}^{(\alpha,\frac{1}{2})}(x,y)$$

where $c = 3\sqrt{3}\pi c_0$.

i.e. we get the same result as Borodin but for a much larger class of external fields V.

Approach

- Relate the correlation kernel to a type II multiple orthogonal polynomial ensemble.
- Relate the type II MOP ensemble to a Riemann-Hilbert problem.
- Apply the **Deift-Zhou method of steepest descent** to the RHP.
- Find and solve a vector equilibrium problem in order to normalize the RHP.
- Find out how Meijer G-functions enter the solution of the (local parametrix) RHP.

The type II multiple orthogonal polynomial ensemble (for n even)

The ensemble can be expressed in terms of polynomials p_j, q_j as

$$K_{V,n}^{\alpha,\frac{1}{2}}(x,y) = w(y)\sum_{j=0}^{n-1}p_j(x)q_j(\sqrt{y}),$$

where in particular

$$\int_0^\infty p_n(x)x^k w(x)dx = \int_0^\infty p_n(x)x^k\sqrt{x}w(x)dx = 0, \qquad k = 0,1,\ldots,\tfrac{n}{2}-1.$$

The Riemann-Hilbert Problem (for n even)

RH-Y1 $Y : \mathbb{C}\setminus[0,\infty) \to \mathbb{C}^{3\times 3}$ is analytic.

RH-Y2 Y has boundary values for $x \in (0,\infty)$, and

$$Y_+(x) = Y_-(x)\begin{pmatrix} 1 & w(x) & \sqrt{x}w(x) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

RH-Y3 As $|z| \to \infty$

$$Y(z) = \left(\mathbb{I} + \mathcal{O}\left(\frac{1}{z}\right)\right)\begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-\frac{n}{2}} & 0 \\ 0 & 0 & z^{-\frac{n}{2}} \end{pmatrix}$$

RH-Y4 As $z \to 0$

$$Y(z) = \mathcal{O}\begin{pmatrix} 1 & h_\alpha(z) & h_{\alpha+\frac{1}{2}}(z) \\ 1 & h_\alpha(z) & h_{\alpha+\frac{1}{2}}(z) \\ 1 & h_\alpha(z) & h_{\alpha+\frac{1}{2}}(z) \end{pmatrix}$$

where $h_\alpha(z)$ equals z^α, $\log z$ and 1 when $\alpha < 0, \alpha = 0$ and $\alpha > 0$ respectively.

A new iterative method to get the matching

We have a local parametrix P at $z = 0$ on a disc D and a global parametrix N such that for $z \in \partial D$

$$P(z)N(z)^{-1} = \mathbb{I} + \frac{A_n(z)}{z} + \mathcal{O}\left(\frac{1}{n}\right), \qquad n \to \infty.$$

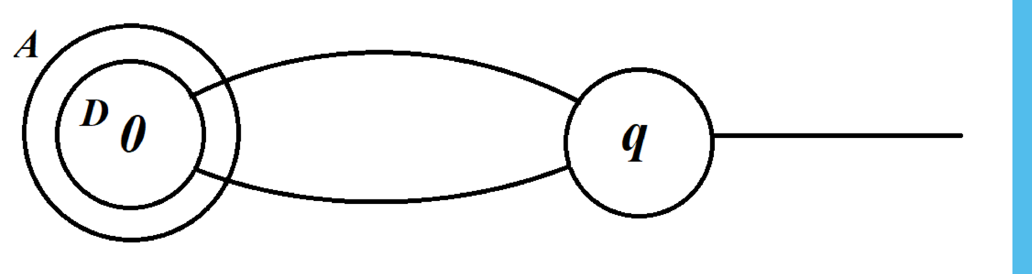
We transform the RHP by adding an annulus A around the disc D and applying the transformation

$$P_D(z) = \left(\mathbb{I} + \frac{A_n(0) - A_n(z)}{z}\right)P(z) \qquad z \in D$$

$$P_A(z) = \left(\mathbb{I} + \frac{A_n(0)}{z}\right)^{-1}N(z) \qquad z \in A.$$

This improves the jump between the local and global parametrix (on $\partial A \setminus \partial D$).

Iterating this process yields the matching condition.



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