The model

The Muttilab-Borodin ensemble [2] with parameter $\theta > 0$ and weight function $w$ is the following probability density function:

$$\frac{1}{Z} \prod_{j<k} (x_j - x_k)(x_j^\theta - x_k^\theta) \prod_{j=1}^n w(x_j).$$

We consider an $n$-dependent weight function $w(x) = x^\alpha e^{-F(x)}$ with $\alpha > -1$ and an external field $V$. The ensemble is a determinantal point process, it can be written as

$$\det \left( K_{\nu,\theta}^n(x_i, x_j) \right)_{1 \leq i, j \leq n}$$

where $K_{\nu,\theta}^n(x, y)$ is the so-called correlation kernel.

Known result and main interest

Our main interest is to study the large $n$ behavior of $K_{\nu,\theta}^n(x, y)$, Borodin [2] computed the hard edge scaling limit for the Laguerre case, namely if $V(x) = x$, then

$$\lim_{n \to \infty} \frac{1}{n^{1-\theta}} K_{\nu,\theta}^n \left( \frac{x}{n^{1-\theta}}, \frac{y}{n^{1-\theta}} \right) = K^{(0)}(x, y)$$

with limiting correlation kernel

$$K^{(0)}(x, y) = \delta y^n \int_0^1 J_{\alpha-1}(ux) J_{\alpha-1}(uy) \, du$$

where

$$J_{\alpha}(x) = \sum_{j=0}^\infty (-1)^j \frac{(\alpha x)^j}{j! (\alpha + j)!}$$

The same limit turns up in products of random matrices [1,4,5]. From these models and others [7] we know that the limit can be expected to hold for a much larger class of external fields $V$.

Equilibrium problem

There is a corresponding equilibrium problem: minimize

$$\frac{1}{2} \int \left( \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \frac{1}{2} \int \log \frac{1}{|x^\theta - y^\theta|} d\mu(x) d\mu(y) + \int V(x) d\mu(x) \right)$$

among all probability measures $\mu$ on $[0, \infty)$. There exists a unique absolutely continuous solution $\mu_{\nu,\theta}$, called the equilibrium measure [6].

We call the external field $V$ one-cut $\theta$-regular if $\mu_{\nu,\theta}$ is supported on one interval $[0, q)$ for some $q > 0$ with a density that is positive on $[0, q)$ and satisfies

$$\frac{d\mu_{\nu,\theta}(s)}{ds} \begin{cases} \cos \theta \pi (1 + o(1)) & \text{as } s \to 0^+, \\ c_1(q-s)^{\alpha/2}(1+o(1)) & \text{as } s \to q^-. \end{cases}$$

Main Theorem

Let $\alpha > -1$, $\theta = 1/2$, and let $V : [0, \infty) \to \mathbb{R}$ be a real analytic external field that is one-cut $\theta$-regular (as above). Then for $x, y \in [0, \infty)$ we have

$$\lim_{n \to \infty} \frac{1}{n} K_{\nu,\theta}^n \left( \frac{x}{n^{1-\theta}}, \frac{y}{n^{1-\theta}} \right) = K^{(0)}(x, y)$$

where $c = 2\sqrt{2\pi}$. i.e. we get the same result as Borodin but for a much larger class of external fields $V$.

Approach

- Relate the correlation kernel to a type II multiple orthogonal polynomial ensemble.
- Relate the type II MOP ensemble to a Riemann-Hilbert problem.
- Apply the Deift-Zhou method of steepest descent to the RHP.
- Find and solve a vector equilibrium problem in order to normalize the RHP.

Iterating this process yields the matching condition.

The type II multiple orthogonal polynomial ensemble (for $n$ even)

The ensemble can be expressed in terms of polynomials $p_j$, $q_j$ as

$$\frac{1}{Z_n} \prod_{j=0}^{n-1} p_j(x) q_j(\sqrt{y})$$

where in particular

$$\int_0^\infty p_k(x) x^k w(x) \, dx = \int_0^\infty p_k(x) x^{k+1} w(x) \, dx = 0, \quad k = 0, 1, \ldots, n-1.$$