On the persistence probability for Kac polynomials and truncated orthogonal matrices

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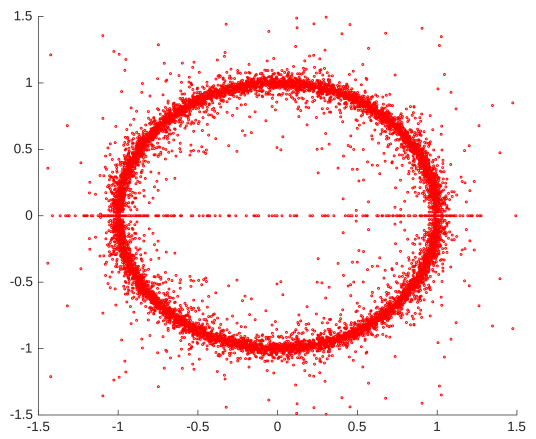
Random Polynomials

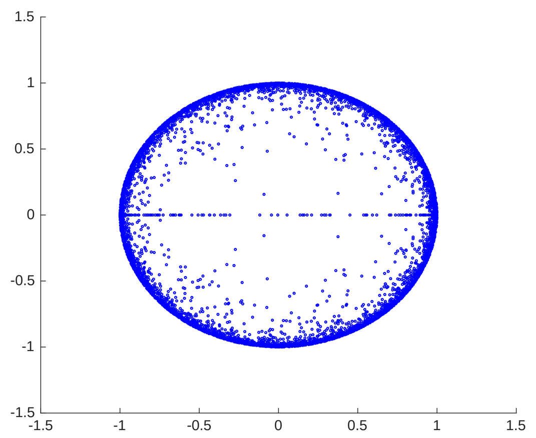
In the area of random polynomials, introduced in late 18^{th} century, one is interested in the distribution of *random* roots. There are many different models of random polynomials and here we study so called *Kac polynomials*

$$K_N(z) = \sum_{k=0}^{N} a_k z^k,$$

where $\{a_k\}_{k=0}^{\infty}$ is a family of i.i.d. mean zero, unit variance random variables having distribution ξ. It was shown before that under some mild conditions on probability distribution ξ, normalized counting measure of zeros converge to the uniform distribution on the unit circle when degree of polynomial $N \to \infty$. However for every single realization of the polynomial it is clear (see Fig. 1a) that there are many real roots.



(a) Zeros of $K_{50}(z)$



(b) Eigenvalues of M_{50}

· *How many roots are real?* Let $\mathcal{N}_N$ denote number of real roots for the polynomial K_N. Then when $N \to \infty$

$$\mathcal{N}_N \sim \mathcal{N}\left(\frac{2}{\pi}\log N,\ \frac{4}{\pi}\left(1-\frac{2}{\pi}\right)\log N\right).$$

· *Is it possible to have no real roots at all?* N is assumed to be even. Heuristically, we penalize removal of every real point by multiplying the probability by $p < 1$. Thus to remove all $O(\log N)$ of them we should multiply by $p^{\log N} = N^{-\log p^{-1}}$. **Power law decay!** of persistence probability

$$p_{2n} := \mathbb{P}\{K_{2n}(x) > 0, \forall x \in \mathbb{R}\} = \frac{1}{2}\mathbb{P}\{K_{2n}(z) \text{ has no real roots}\}.$$

Random series

One can study random series instead of polynomials, i.e.

$$K(z) = \sum_{k=0}^{\infty} a_k Z^k.$$

Advantages: **Pfaffian Point Process** formed by zeros.
Disadvantages: No N left and complicated formulas.
Theorem[Matsumoto, Shirai '13] *Real zeros of Gaussian random series inside the unit disk form a PPP with the kernel*

$$\mathcal{K}(x,y) = \begin{pmatrix} \partial_x\partial_y k(x,y) & \partial_x k(x,y) \\ \partial_y k(x,y) & k(x,y) - \alpha\varepsilon(x,y) \end{pmatrix}, \qquad (1)$$

where $\alpha = -1$, $\varepsilon(x,y) = \frac{1}{2}\operatorname{sgn}(y-x)$,

$$k(x,y) = \frac{\operatorname{sgn}(y-x)}{\pi}\arcsin\frac{\sqrt{1-x^2}\sqrt{1-y^2}}{1-xy} - \varepsilon(x,y).$$

Partial analysis was announced by Fitzgerald, Tribe, Zaboronski.

Pfaffians are everywhere

GSP with sech correlator

Theorem[Dembo, Poonen, Shao, Zeitouni '00] *Let* $\{Y_t\}_{t\geq 0}$ *be a GSP with correlator* $R(t) = \operatorname{sech}\frac{t}{2}$. *Then under some mild conditions on the distribution* ξ *one has*

$$p_{2n} \sim n^{-4\theta}, \quad \theta = -\lim_{T\to\infty}\frac{1}{T}\log\left[Y_t > 0, t \in [0,T]\right].$$

Heuristics Change $x = 1 - \mathrm{e}^{-t}$, $a_i = W_{i+1} - W_i$, where W_t is a BM, then $K_{2n}(x) \sim \int_0^\infty \mathrm{e}^{-u\mathrm{e}^{-t}}\mathrm{d}W_u := \hat{Y}_t$. And for $Y_t = \hat{Y}_t/\sqrt{\mathrm{Var}\hat{Y}_t}$ one has $\mathbb{E}\{Y_tY_s\} = \operatorname{sech}\frac{t-s}{2}$.
No GSP with explicitly known persistence constant...
Luckily for GSP with sech correlator we have **Pfaffian structure!**
Theorem[1] *Zeros of GSP with* sech *correlator form a Pfaffian Point Process with the kernel* (1) *where* $\alpha = -1$ *and*

$$k(x,y) = \frac{2}{\pi}\arctan\tanh\frac{x-y}{4}.$$

Pfaffian structure (1) allows one to use Tracy-Widom calculation:

$$\mathbb{P}\{\text{no points inside } [0,T]\} =: p_T = \sqrt{\det(\mathrm{I} - \mathcal{C}\chi_T)\,\Gamma_T},$$

where $\mathcal{C}\chi_T$ is a Wiener-Hopf operator with the kernel $c(x) = \frac{1}{2\pi}\operatorname{sech}\frac{x}{2}$ truncated to an interval $[0,T]$ and

$$\Gamma_T = 1 + \int_{-\infty}^{0}\left(R(x,0) - R(x,T)\right)\mathrm{d}x,$$

where $R(x,y)$ is a kernel of the resolvent operator $(\mathrm{I} - \mathcal{C}\chi_T) - \mathrm{I}$. Using modification of Akhiezer-Kac-Szego theorem we get

$$\det(\mathrm{I} - \mathcal{C}\chi_T) = \mathrm{e}^{-3T/8}\frac{2^{5/12}\pi}{\Gamma^2\left(\frac{1}{4}\right)}\mathrm{e}^{6\zeta'(-1)}\left(T + 6\log 2 + O\left(\mathrm{e}^{-T/2}\right)\right).$$

For Γ_T we can prove only [2] $\Gamma_T = \mathrm{e}^{o(T)}$ which is enough for calculating persistence constant but we are working now on getting next order corrections to persistence probability.

Truncations of orthogonal matrices

Consider ensemble of $2n \times 2n$ random matrices M_{2n} obtained from random orthogonal matrices of size $(2n+\ell)\times(2n+\ell)$ by removing last ℓ columns and rows. Eigenvalues rank-one truncations (see Fig. 1b) are distributed similarly to random roots of Kac polynomials restricted to the interior of the unit disk.
Lemma[Krishnapur '09 - Forrester '10] If $O = \begin{pmatrix} M_{2n} & \mathbf{u} \\ \mathbf{v}^T & a \end{pmatrix}$, then
$\frac{\det(z - M_{2n})}{\det(\mathrm{I} - zM_{2n})} = a + z\mathbf{v}^T(\mathbb{I} - zM_{2n})^{-1}\mathbf{u} = a + \sum_{k=1}^{\infty}\mathbf{v}^T M_{2n}^{k-1}\mathbf{u}z^k.$

Joint eigenvalues distribution conditioned to have L real ones and M pairs o complex conjugate ones is given by

$$p_\ell^{(L,M)}\left(\vec{\lambda},\vec{Z}\right) = 2^M C_{N,\ell}\left|\Delta\left(\vec{\lambda}\cup\vec{Z}\right)\right|\prod_{j=1}^{L}w(\lambda_j)\prod_{j=1}^{2M}w(z_j),$$

where

$$w^2(z) = \begin{cases} \frac{1}{2\pi}|1-z|^{-1}, & \ell = 1, \\ \frac{\ell(\ell-1)}{2\pi}|1-z^2|^{\ell-2}\int_{\frac{2|\Im z|}{|1-z^2|}}^{1}(1-u^2)^{\frac{\ell-3}{2}}\mathrm{d}u, & \ell \geq 2. \end{cases}$$

This form of the distribution yields Pfaffian structure:

$$\mathbb{E}\left\{\prod_{i=1}^{2n}f(\zeta_i)\right\} = \frac{\mathrm{Pf}\ (U_f)}{\mathrm{Pf}\ (U_1)},$$

where U_f is a skew symmetric matrix of size $2n\times 2n$ with entries $u_{j,k} = (p_{j-1}f, p_{k-1}f)_w$ and skew product depending on w.
Choosing $f(z) = 1 - (1-\mathrm{e}^s)\chi_{\mathbb{R}}(z)$ and appropriate family of skew-orthogonal polynomials we obtain for $\ell = 1$ [3]

$$\mathbb{E}\left\{\mathrm{e}^{s\mathcal{N}_{2n}}\right\} = \det\left\{\mathrm{I} - \frac{1-\mathrm{e}^{2s}}{\pi(j+k+1/2)}\right\}_{j,k=0}^{n-1} \sim n^{-\Psi(s)},$$

$$\psi(s) = \frac{1}{8} - \left[\frac{\sqrt{2}}{\pi}\cos^{-1}\left(\frac{\mathrm{e}^s}{\sqrt{2}}\right)\right]^2$$

Main result

Persistence probability p_{2n} for Kac polynomials and q_{2n} for one-rank truncations of random orthogonal matrices decay as

$$\boxed{p_{2n} \sim n^{-3/4}, q_{2n} \sim n^{-3/8}, \quad n \to \infty.}$$

Persistence probability for semi-infinite Potts model

We consider a semi-infinite Ising spin chain, whose configuration at time t is given by $\{\sigma_i(t)\}_{i\geq 0}$, with $\sigma_i(t) = \pm 1$. Initially, the system is in a random initial configuration where $\sigma_i(0) = \pm 1$ with equal probability 1/2 and, at subsequent time, the system evolves according to the Glauber dynamics. Within each infinitesimal time interval Δt, every spin is updated according to

$$\sigma_i(t+\Delta t) = \begin{cases} \sigma_i(t), & \text{with proba. } 1 - 2\Delta t, \\ \sigma_{i-1}(t), & \text{with proba. } \Delta t, \\ \sigma_{i+1}(t), & \text{with proba. } \Delta t, \end{cases}$$

Ising model can be mapped to coalescing random walks by tracing back in time the value of the spin i at time t. This mapping allowed us to show
Theorem[1] $\langle\sigma_0(t_1)\cdots\sigma_0(t_{2m})\rangle \sim \mathrm{Pf}(A)$, *with* $a_{i,j} = \operatorname{sgn}(j-i)\langle\sigma_0(t_i)\sigma_0(t_j)\rangle = \frac{2}{\pi}\arcsin\frac{2\sqrt{t_it_j}}{t_i+t_j}$.
Instead of having 2 states ± 1 let us now study q-states Potts model. Then the probability of the spin at origin not flipping up to time T has an asymptotic behaviour of the form

$$p_T \sim T^{-\theta(q)}, \quad \theta(q) = \frac{1}{8} - \left[\frac{\sqrt{2}}{\pi}\arccos\frac{2-q}{q\sqrt{2}}\right]^2.$$

Diffusion equation with

Let $\mathbf{x} \in \mathbb{R}^d$ and $\phi : \mathbb{R}^d \times \mathbb{R}_+$ satisfy

$$\begin{cases} \partial_t\phi(\mathbf{x},t) = \Delta_{\mathbf{x}}\phi(\mathbf{x},t), \\ \phi(\mathbf{x},0) = \Psi(\mathbf{x}), \end{cases}$$

where $\Psi(\cdot)$ is a Gaussian delta correlated random field. Then

$$\phi(\mathbf{x},t) = \int G(\mathbf{x}-\mathbf{y},t)\Psi(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

with $G(\mathbf{x},t)$ being Green's function and

$$\mathbb{E}\{\phi(\mathbf{x},t)\phi(\mathbf{x},s)\} = \operatorname{sech}^{d/2}\frac{t-s}{2}.$$

Discussion & Open problems

- Are there other examples of Gaussian Stationary Process with zeros forming Pfaffian Point Process? What are analytical properties of corresponding correlators? - *first results in Kabluchko'18.*
- Asymptotic analysis of $q_{n,\ell}$ when $n \to \infty$ and $1 \ll \ell \ll n$; $\ell \sim n$; $\ell \gg n$.
- Distribution of the largest real root for Kac polynomials or the closest to zero real eigenvalue for truncations of orthogonal matrices? - *largest real root is heavy-tailed by Butez '17.*
- Direct connection between persistence probability in q-states Potts model and the Laplace transform of the number of real eigenvalues - *we have it only on the level of answers [3].*
- 'Higher dimensional' GSP with $\operatorname{sech}^{d/2}\frac{t}{2}$ and corresponding persistence probability - *some numerics exist and they show obvious growth with d.*

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