Microwave graphs:
Transition from Symplectic to Orthogonal Symmetry

Martin Richter[1,2], Aimaiti Rehemanjiang[3], Ulrich Kuhl[1], Hans-Jürgen Stöckmann[3]

[1]Université Côte d'Azur, CNRS, InPhyNi, Nice, France

[2]School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, UK

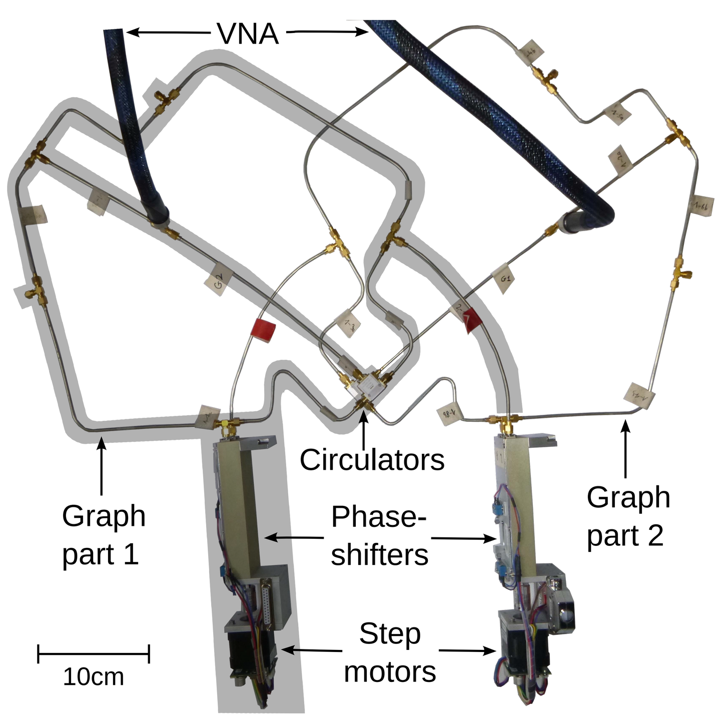
[3]Fachbereich Physik der Philipps-Universität Marburg, D-35032 Marburg, Germany

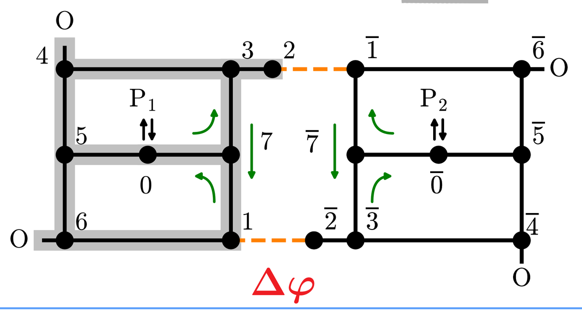
Introduction

A. Rehemanjiang, M. Allgaier, C. H. Joyner, S. Müller, M. Sieber, U. Kuhl, H.-J. Stöckmann *PRL* **117**, 064101 (2016)

A. Rehemanjiang, M. Richter, U. Kuhl, H.-J. Stöckmann *PRE* **97**, 022204 (2018)

- Experimental Realization of GSE Spectra

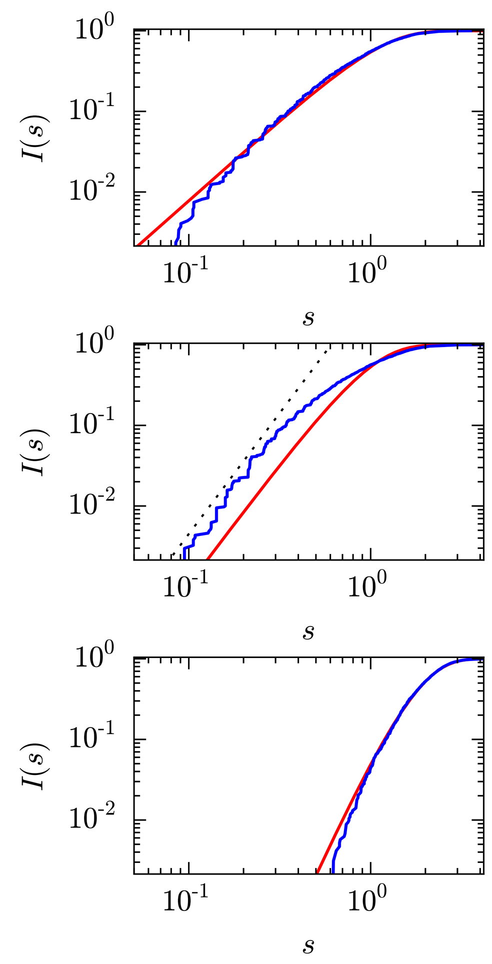




- Generate two *Complex Conjugate* Subgraphs
- Add *Inversion Symmetry*

Transition GSE $\longrightarrow$ GUE $\longrightarrow$ GOE

- Integrated next-nearest level-spacing distribution
$$I(s) = \int_0^s \mathrm{d}s' p(s') \sim s^{\beta+1}$$



$\Delta\varphi$

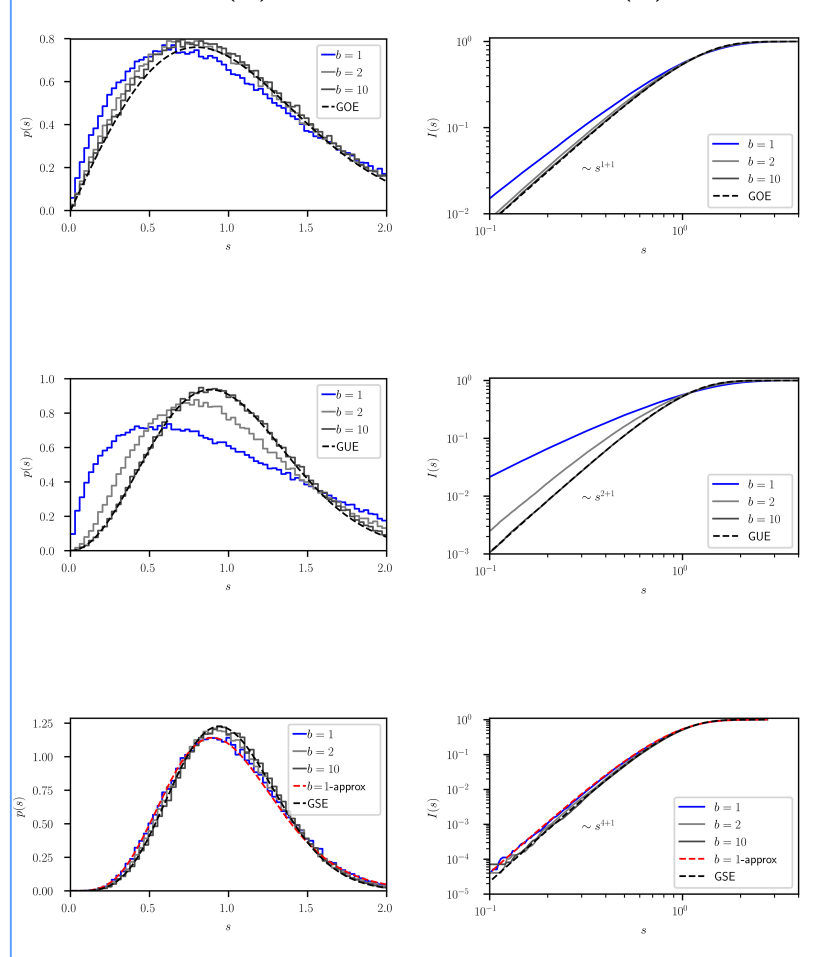
2π GOE

$3\pi/2$ GUE ?

π GSE

Note: Symmetries make GSE and GOE more robust and easier to measure. GUE has no symmetry but is it mixed enough?

Transition for Integrated Spacings



$\Delta\varphi$

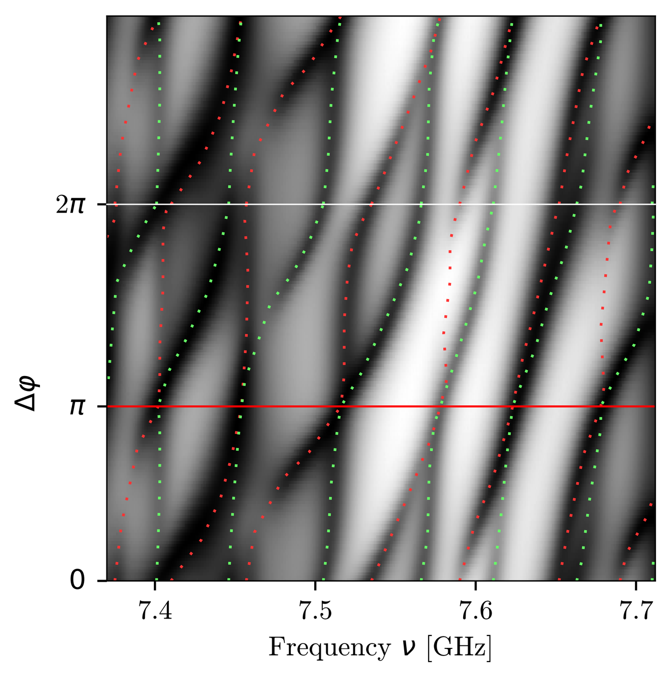
2π GOE $I(s) \sim s^{1+1}$

$3\pi/2$ GUE $I(s) \sim s^{2+1}$

π GSE $I(s) \sim s^{4+1}$

Spectra

- Depend on Phase: *Kramers Degeneracy* for $\Delta\varphi = \pi$



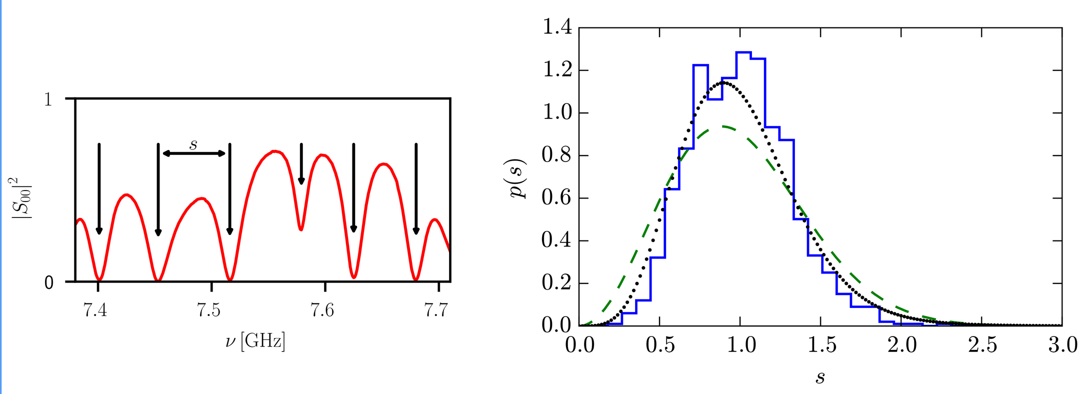
$\Delta\varphi$

2π GOE

$3\pi/2$ GUE ?

π GSE

- *Next-Nearest Level Spacings*



Theory for Two Bonds: 2×2 Model

- Conjugate GUE blocks $\rightarrow$
  - *Two-level* approximation
  - Representation in *Eigenenergies*: $H_0 = \begin{pmatrix} a/2 & \\ & -a/2 \end{pmatrix}$
  - GUE Spacing $p(a) = \frac{32}{\pi^2} a^2 \mathrm{e}^{-\frac{4}{\pi}a^2}$
- coupled by $b = 2$ pairs of bonds $\rightarrow$
- with $\Delta\varphi = \pi$
  - $\left(VV^\dagger\right) = \begin{pmatrix} & -V_{12} \\ V_{12} & \end{pmatrix}$

    $V_{12} = \psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2)$

    $\psi_i(x_j)$: complex Gaussian

    $p(\psi_R, \psi_I) = \mathrm{e}^{-\pi|\psi|^2}$

Distribution of Nearest Neighbour Spacings

$$p_0(s) = 16 s^4 \int_0^\pi \mathrm{d}\varphi \sin\varphi \cos^2\varphi \, \mathrm{e}^{-\frac{4}{\pi}s^2\cos^2\varphi} \hat{K}_1(\pi s \sin\varphi)$$

with

$$\hat{K}_1(t) = t K_1(t), \quad \lim_{t \to 0} \hat{K}_1(t) = 0$$

and K_1: modified Bessel function.

Position Space vs. Eigenenergies

Necessary for scattering systems: Keep *inversion symmetry*
$\rightarrow$ Attach channels to symmetric points (Position space)

With $H_0 = \Psi^\dagger \Lambda \Psi$ and $\Lambda = \mathrm{diag}(E_1, E_2, \dots)$:

$$H = \begin{pmatrix} \Lambda & VV^\dagger \\ VV^\dagger & \Lambda \end{pmatrix} = \begin{pmatrix} \Psi H_0 \Psi^\dagger & VV^\dagger \\ VV^\dagger & \Psi^* H_0^* \Psi^T \end{pmatrix}$$
$$= \begin{pmatrix} \Psi & 0 \\ 0 & \Psi^* \end{pmatrix} \begin{pmatrix} H_0 & WW^\dagger \\ WW^\dagger & H_0^* \end{pmatrix} \begin{pmatrix} \Psi & 0 \\ 0 & \Psi^* \end{pmatrix}^\dagger$$

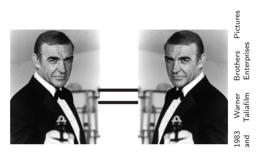
$$\left(\Psi^\dagger VV^\dagger \Psi^*\right)_{ij} = \left(WW^\dagger\right)_{ij}$$
$$= \frac{N}{\pi} \sum_{k=1}^{b} \left(\delta_{2k-1,i}\delta_{2k,j} + \mathrm{e}^{\mathrm{i}\Delta\varphi}\delta_{2k,i}\delta_{2k-1,j}\right)$$

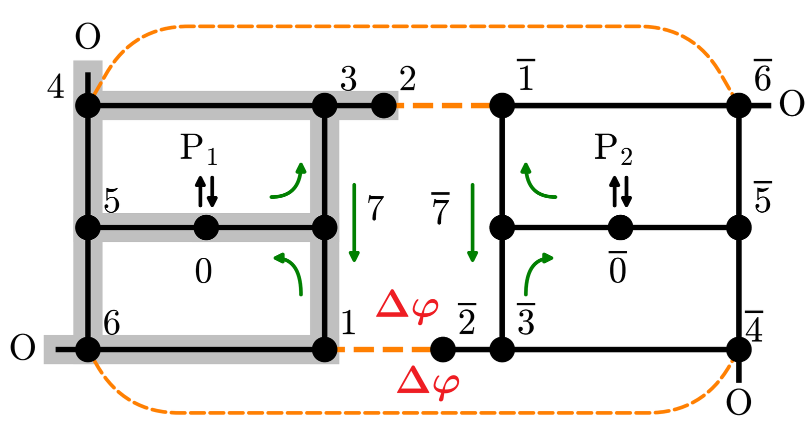
$$WW^\dagger = \frac{N}{\pi} \left.\begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\Delta\varphi} & & & \\ & & 1 & \mathrm{e}^{\mathrm{i}\Delta\varphi} & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix}\right\} b = 2$$

Model and Symmetries

- Hamiltonian: *Block* structure
  Complex Conjugate Subsystems
  Coupled by **one** Pair of Bonds (having $\Delta\varphi$)
$$H = \begin{pmatrix} H_0 & VV^\dagger \\ VV^\dagger & H_0^* \end{pmatrix}$$
- *Coupling* between Subsystems:
$$\left(VV^\dagger\right)_{nm} = \psi_{n1}\psi_{m\bar{2}} + \mathrm{e}^{\mathrm{i}\Delta\varphi}\psi_{n2}\psi_{m\bar{1}}$$
- *Symmetries*: Reorder
$$(1, 2, 3, \dots, \bar{1}, \bar{2}, \bar{3}, \dots)$$
$$\downarrow$$
$$(1, \bar{1}, 2, \bar{2}, 3, \bar{3}, \dots)$$
- GSE: $\Delta\varphi = \pi$ $\quad [H, \mathcal{T}] = 0, \mathcal{T}^2 = -1$
$$\mathcal{T} = \mathrm{diag}(\mathcal{C}\tau_y, \mathcal{C}\tau_y \dots) \text{ with } \tau_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
- GOE: $\Delta\varphi = 2\pi$ $\quad [H, \mathcal{T}] = 0, \mathcal{T}^2 = +1$
$$\mathcal{T} = \mathrm{diag}(\mathcal{C}\tau_x, \mathcal{C}\tau_x \dots) \text{ with } \tau_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Additional Pairs of Bonds





- Circulators
- Phase shifters
- b bond pairs

$$\left(VV^\dagger\right)_{nm} = \psi_{n1}\psi_{m\bar{2}} + \mathrm{e}^{\mathrm{i}\Delta\varphi}\psi_{n2}\psi_{m\bar{1}}$$
$$= \sum_{i=1}^{b} \left(\psi_{n,2i-1}\psi_{m,\overline{2i}} + \mathrm{e}^{\mathrm{i}\Delta\varphi}\psi_{n,2i}\psi_{m,\overline{2i-1}}\right)$$

Outlook

- Scattering Properties of TRS systems
  i.e. What is about $S_{12} = S_{21}^*$ in case of GOE?
  Necessary: Keep *inversion symmetry* when adding channels.
- Analyse influence of bond pairs in *Scattering Setups*

  See Also:

A. M. Martínez-Argüello, A. Rehemanjiang, M. Martínez-Mares J. A. Méndez-Bermúdez, H.-J. Stöckmann, H.-J., U. Kuhl, *PRB* **98**, 075311 (2018)

- Analyse *Level Dynamics* $\lambda(\Delta\varphi)$ for various numbers of bonds b.
  (Compare *global* vs. *local* perturbations in billiards)
- Distribution of Reflections