Eigenvalue Distribution of a Quasi-Hermitian Random Matrix Model

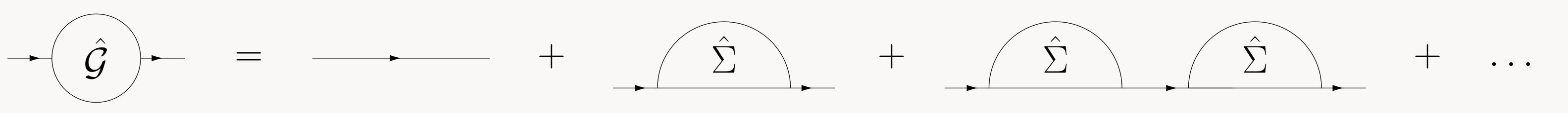
Roman Riser (Department of Mathematics, University of Haifa, Israel)
joint work with Joshua Feinberg

Definitions

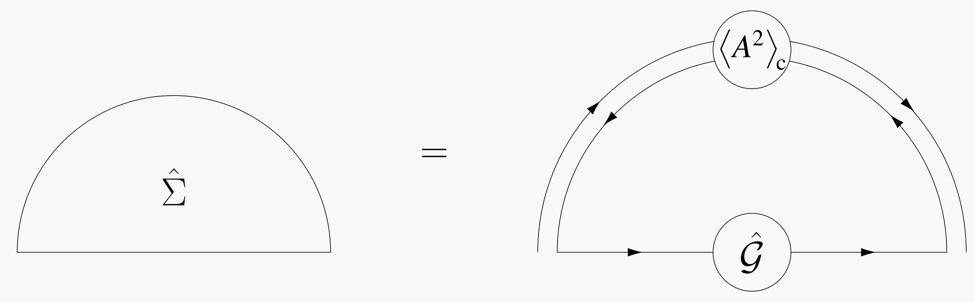
- **Quasi-hermiticity.** Let B and φ be $N \times N$ matrices where φ is a complex matrix and B a Hermitian, positive metric. We call φ (strictly) quasi-Hermitian with respect to the metric B if it fulfills the *intertwining relation*
$$\varphi^\dagger B = B\varphi, \tag{1}$$
i.e. if φ is Hermitian with respect to the metric B. For $B = \mathbb{1}$ it reduces to ordinary hermiticity.
- The general solution of the intertwining relation (1) is
$$\varphi = AB, \tag{2}$$
where $A = A^\dagger$ is an arbitrary Hermitian matrix. [1] Thus, given B, there are N^2 independent quasi-Hermitian matrices with respect to B.
- For a positive metric, the spectrum of φ is real and has been studied in [1,2,3].
- **Indefinite metric.** In the following we relax the condition that B has to be positive-definite and mainly concentrate on the indefinite metric when B is of the form
$$B = \mathrm{diag}(\underbrace{1,\dots,1}_{k},\underbrace{-1,\dots,-1}_{N-k}). \tag{3}$$
- Equation (2) still describes the general solution of the intertwining relation (1). Though the spectrum no longer has to be purely real but may contain pairs of complex-conjugated eigenvalues.
- We will analyze the spectrum of φ when B is deterministic and A is randomly chosen from the GUE ensemble, i.e. has the probability distribution
$$P(A) = Z_N^{-1} \mathrm{e}^{-\frac{Nm^2}{2}\mathrm{Tr}A^2},$$
where $m > 0$ is a parameter and Z_N the normalization constant. This induces a probability distribution on φ,
$$P(\varphi) = \tilde{Z}_N^{-1} \mathrm{e}^{-\frac{Nm^2}{2}\mathrm{Tr}(B^{-2}\varphi^\dagger\varphi)} = \tilde{Z}_N^{-1} \mathrm{e}^{-\frac{Nm^2}{2}\mathrm{Tr}(\varphi^\dagger\varphi)}.$$

Method for analyzing the large N limit: Feynman Diagrams and Hermitization

- In the large N limit we can obtain the averaged eigenvalue density of the quasi-Hermitian matrix φ from the resolvent of $\varphi = AB$ (Green's function),
$$G(w) = \left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{w - AB}\right\rangle, \tag{4}$$
where we average over the GUE ensemble of A.
- Averaging over A becomes easier if one can avoid the product of matrices AB using the trick due to [4] where they introduce the $2N \times 2N$ matrix
$$H = \begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}, \qquad \text{with the non-Hermitian resolvent} \qquad \frac{1}{z-H} = \begin{pmatrix} \frac{z}{z^2-AB} & \frac{1}{z^2-AB}A \\ \frac{1}{z^2-BA}B & \frac{z}{z^2-BA} \end{pmatrix}.$$
From its upper left block we can get the Green's function (4) (dividing by z and substituting $w = z^2$).
- Since the resolvent of H is not Hermitian, the next step uses the method of Hermitization [5,6,7,8] where we end up with the $4N \times 4N$ resolvent
$$\hat{\mathcal{G}} = \left[\begin{pmatrix} \eta & z \\ z^* & \eta \end{pmatrix} - \begin{pmatrix} 0 & H \\ H^\dagger & 0 \end{pmatrix}\right]^{-1} = \begin{pmatrix} \eta & 0 & z & -A \\ 0 & \eta & -B & z \\ z^* & -B & \eta & 0 \\ -A & z^* & 0 & \eta \end{pmatrix}^{-1}.$$
- The following procedure is now to expand the resolvent $\hat{\mathcal{G}}$ in powers of bare propagators $\hat{\mathcal{G}}_0 = \hat{\mathcal{G}}_{|A=0}$ and rearrange the perturbative expansion in terms of the self energy $\hat{\Sigma}$.



We can express $\hat{\Sigma}$ in terms of the connected cumulants of the distribution of A and the full propagator $\langle\hat{\mathcal{G}}\rangle$. In the planar limit we arrive at the following expression for the self energy



$$= \begin{pmatrix} \overline{44} & 0 & 0 & \overline{41} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \overline{14} & 0 & 0 & \overline{11} \end{pmatrix}$$

where $\overline{\alpha\beta} = N^{-1}\langle \mathrm{Tr}\,\hat{\mathcal{G}}_{\alpha\beta}\rangle$ are block traces. After taking $\eta \to 0$, they are uniquely determined by the Schwinger-Dyson equation $\langle\hat{\mathcal{G}}\rangle = \frac{1}{\hat{\mathcal{G}}_0^{-1} - \hat{\Sigma}}$. All other blocks of $\langle\hat{\mathcal{G}}\rangle$ can be found in terms of the four quantities $\overline{11}, \overline{14}, \overline{41}, \overline{44}$. In particular
$$\left\langle \frac{1}{N}\mathrm{Tr}\,\hat{\mathcal{G}}_{31}\right\rangle = \left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{z^2 - AB}\right\rangle = \frac{m^2 z}{N}\mathrm{Tr}\frac{B^{-1}(\overline{14} - m^2 w^* B^{-1})}{\overline{11}\cdot\overline{44} - (\overline{41} - m^2 w B^{-1})(\overline{14} - m^2 w^* B^{-1})}.$$

Holomorphic solution: Eigenvalues on the real line

- If $\overline{11} = \overline{44} = 0$, then $\overline{41}$ is a holomorphic function of w as well as
$$\left\langle \frac{1}{N}\mathrm{Tr}\frac{1}{w - AB}\right\rangle = -\frac{m^2}{N}\mathrm{Tr}\frac{B^{-1}}{\overline{41} - m^2 w B^{-1}}.$$
It accounts for the real part of the spectrum of AB.
- If B is of the form (3), the holomorphic gap equation reduces to the cubic equation
$$m^2 b + \frac{\lambda}{b+w} + \frac{1-\lambda}{b-w} = 0, \qquad \text{with } b = -m^{-2}\overline{41}.$$
- For the line density of real eigenvalues one can find [9]
$$\rho^{(1)}(x) = \frac{1}{2\pi}\lim_{\epsilon\searrow 0}\mathrm{Im}\left[G(x - \mathrm{i}\epsilon) - G(x + \mathrm{i}\epsilon)\right] = \mathrm{sign}(1-2\lambda)\frac{\left|\xi - \sqrt{\Delta}\right|^{2/3} - \left|\xi + \sqrt{\Delta}\right|^{2/3}}{\sqrt{3}\cdot 2^{2/3}\cdot 6\pi m^2 x}, \qquad |x| < |w_\pm|,$$
where the density is supported on $[w_-, w_+]$. $\lambda = k/N$ describes the ratio of ones in B (see (3)),
$$w_\pm = \pm\left(\frac{3|1-2\lambda|^{2/3}\left[\left(1-2\sqrt{\lambda(1-\lambda)}\right)^{1/3} + \left(1+2\sqrt{\lambda(1-\lambda)}\right)^{1/3}\right]+2}{2m^2}\right)^{1/2},$$
$$\xi = \xi(x) = -27m^4(1-2\lambda)x, \qquad \Delta = \Delta(x) = \xi^2 + 4\cdot 27 m^6(1 - m^2x^2)^3.$$
- Integrating $\rho^{(1)}(x)$ one finds that the fraction of real eigenvalues is given by $|1-2\lambda|$. In [10] it has been proven algebraically that $|1-2\lambda|$ gives a lower bound for this fraction, even for finite N.

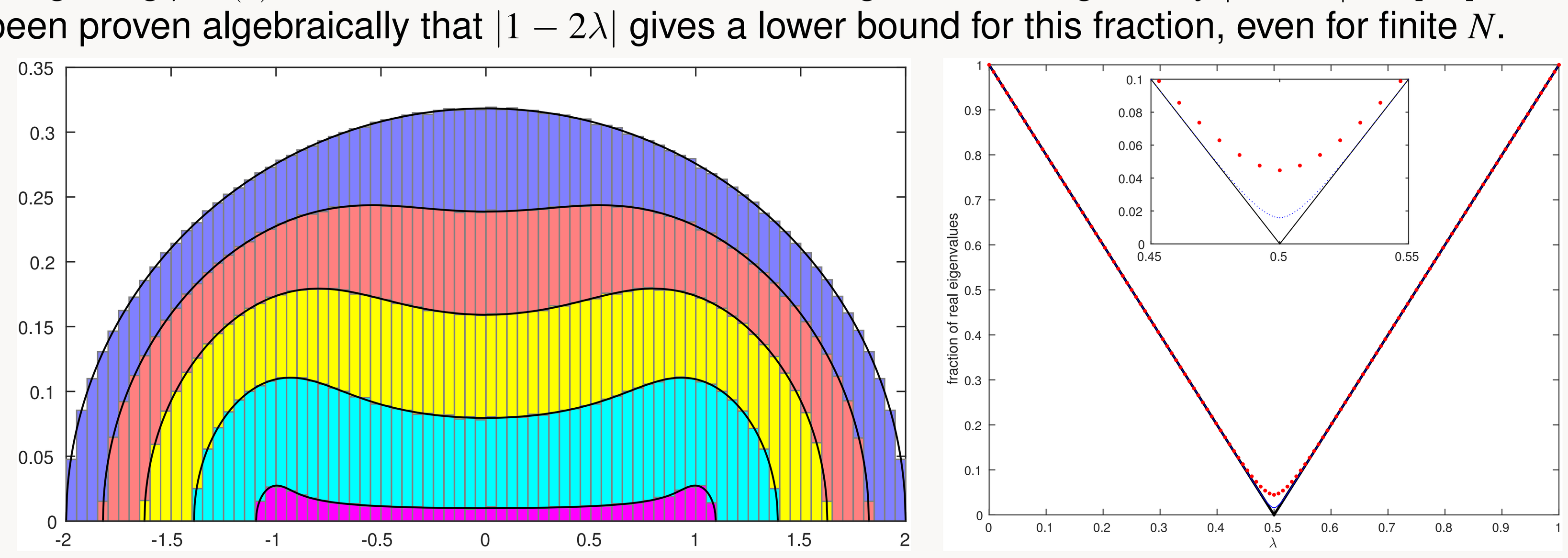


Figure: (left) Histogram of real eigenvalues from a single sample using a numerical simulation with $N = 32768$. Different color represent different λ: 31/64, 3/8, 1/4, 1/8, 1/32768 (ordered as density increases). The solid black line shows the theoretical prediction. (right) Fraction of real eigenvalues as a function of λ. Solid black line shows the theoretical prediction, while the dots represent numerical simulations with $N = 128$ averaged over 500000 samples (red) and $N = 1024$ using 2000 samples (blue).

Non-holomorphic solution: Domain of complex eigenvalues

- $\overline{11}$ and $\overline{44}$ are either both zero or non-zero. If $\overline{11} = \overline{44} \neq 0$, all quantities are non-holomorphic functions of w. It accounts for eigenvalues of AB in the complex plane. The boundary of the two-dimensional domain occupied by these eigenvalues is obtained by setting $\overline{11}$ and $\overline{44}$ to zero in the non-holomorphic gap equations.
- For any positive-definite B, one can consistently show that only the holomorphic solution exists.
- If B is of the form (3), the holomorphic and non-holomorphic solution co-exist for $0 < \lambda = k/N < 1$. The boundary of these blobs can be parametrized in polar coordinates as [9]
$$r_\pm(\theta) = \frac{1}{\sqrt{2}m}\left(1 \pm \sqrt{1 - \left(\frac{\sin\theta_0}{\sin\theta}\right)^2}\right)^{1/2}, \qquad \text{when } \sin^2\theta \geq \sin^2\theta_0 \text{ with } \sin\theta_0 = |2\lambda - 1|. \tag{5}$$
r_+ is the part of the boundary farther from the origin, and r_- the closer one.

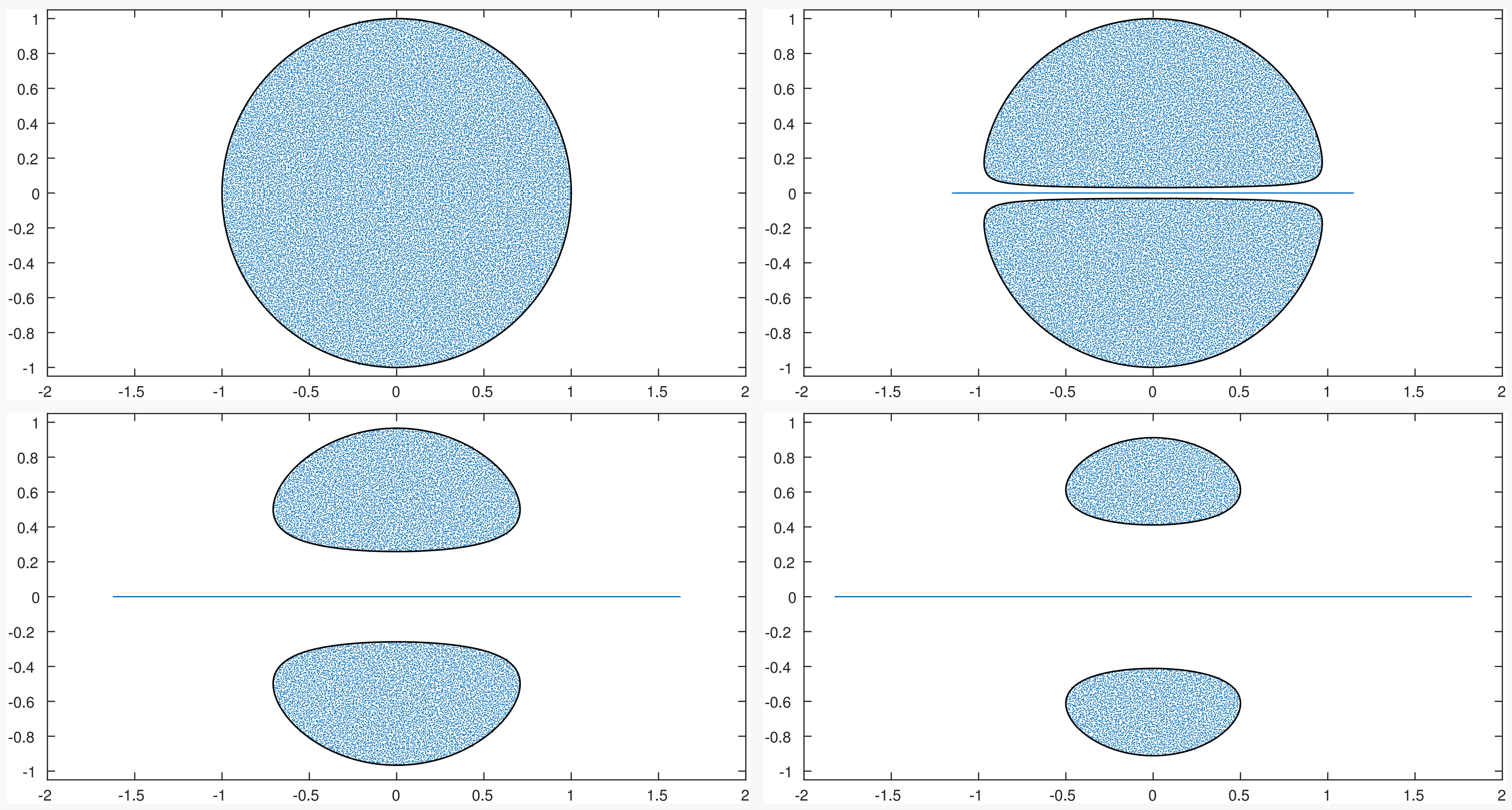


Figure: The blue dots represent the eigenvalues of AB obtained from numerical simulations using a single realization with $N = 32768$ for various λ: 1/2, 15/32, 1/4, 1/8 (from upper left to lower right). The solid black line shows the theoretical boundary prediction given by (5).

- The blobs are symmetric regarding the real axis and the model is symmetric under the transformation $\lambda \mapsto 1 - \lambda$.
- The distance of the blob from the real axis is $m^{-1}\sin(\theta_0/2)$ and touches the line when λ approaches $1/2$. This is a special case where there is no line density of real eigenvalues and the complex eigenvalues uniformly fill a disk, like in the Ginibre ensemble.
- The area of the two blobs we obtain by
$$2\int_{r_-}^{r_+}\int_{\theta_0-\pi}^{2\pi-\theta_0} r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_{\theta_0-\pi}^{2\pi-\theta_0}\left(r_+^2(\theta) - r_-^2(\theta)\right)\mathrm{d}\varphi = \frac{2\pi\lambda}{m^2}.$$
- From the Green's function we can also get the density inside the blob which turns out to be constant and independent of λ,
$$\rho(w) = \frac{1}{\pi}\frac{\partial}{\partial w^*}G(w) = \frac{m^2}{\pi}.$$

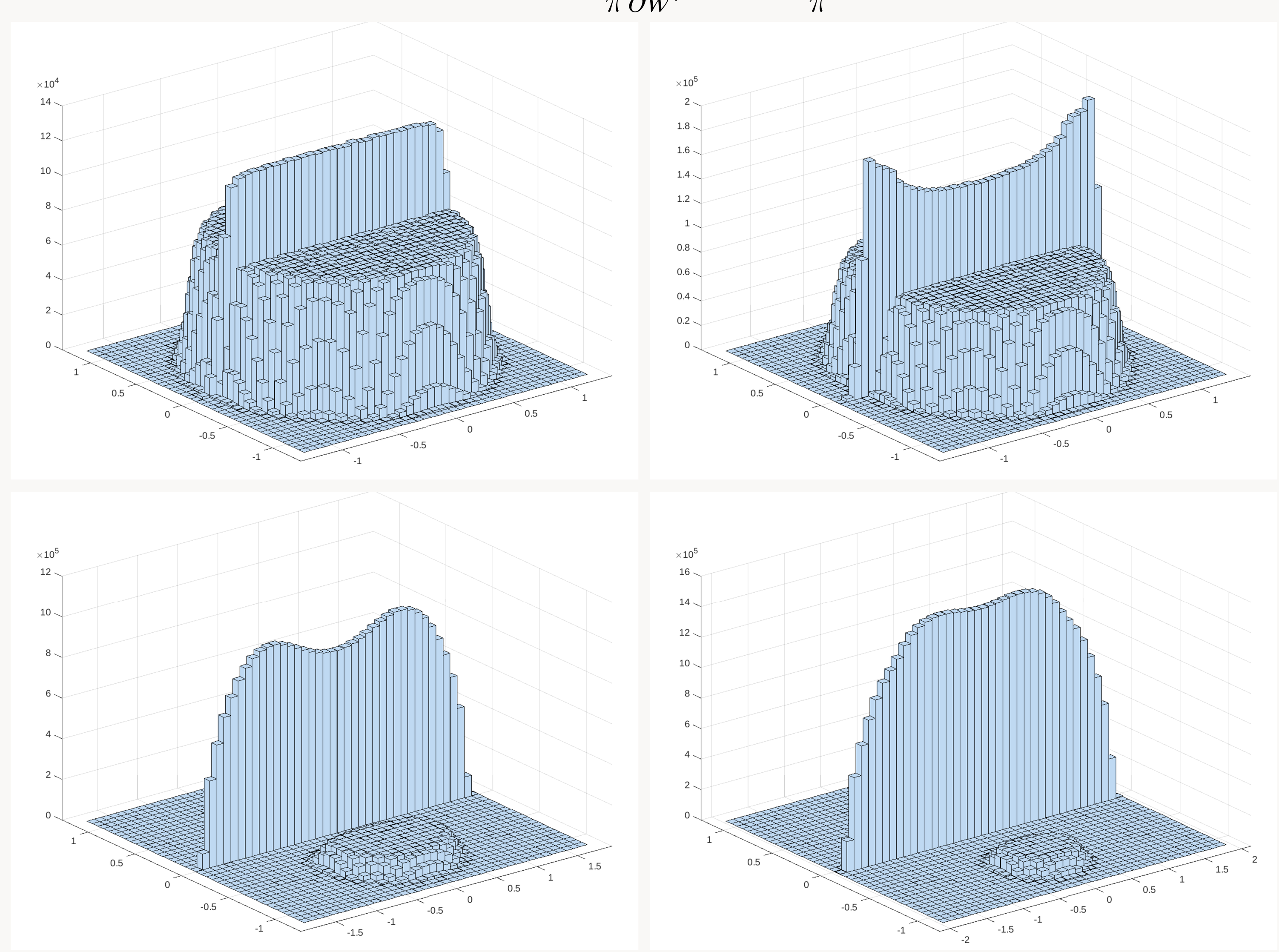


Figure: Histograms with two-dimensional bins in the complex plane showing the eigenvalue distribution obtained by numerical simulations of 500000 samples with $N = 128$ for various λ: 1/2, 15/32, 1/4, 1/8 (from upper left to lower right). For finite N there can be additional real eigenvalues as one notices from the figure in the section holomorphic solution.

References

[1] Y. N. Joglekar, W. A. Karr. Level density and level-spacing distributions of random, self-adjoint, non-Hermitian matrices. *Phys. Rev. E* 83 (2011) 031122

[2] T. Deguchi, P. K. Ghosh, K. Kudo. Level statistics of a pseudo-Hermitian Dicke model. *Phys. Rev. E* 80 (2009) 026213.

[3] S. Kumar, Z. Ahmed. Pseudosymmetric random matrices: Semi-Poisson and sub-Wigner statistics. *Phys. Rev. E* 96 (2017) 022157.

[4] Z. Burda, R. A. Janik, M. A. Nowak. Multiplication law and S transform for non-hermitian random matrices. *Phys. Rev. E* 84 (2011) 061125

[5] J. Feinberg, A. Zee. Non-hermitian random matrix theory: Method of hermitian reduction. *Nuclear Physics B* 504 (1997): 579-608.

[6] R. A. Janik, M. A. Nowak, G. Papp, I. Zahed. Non-hermitian random matrix models. *Nuclear Physics B* 501 (1997): 603-642.

[7] J. Chalker, Z. J. Wang. Diffusion in a random velocity field: Spectral properties of a non-hermitian Fokker-Planck operator. *Phys. Rev. Lett.* 79 (1997): 1797.

[8] K. B. Efetov. Directed quantum chaos. *Phys. Rev. Lett.* 79 (1997): 491.

[9] J. Feinberg, R. Riser. Density of eigenvalues in a quasi-hermitian random matrix model: The case of indefinite metric. *to appear*

[10] D. H. Carlson. On real eigenvalues of complex matrices. *Pacific J. Math.* 15 (1965): 1119-1129.

The author is thankful for the support by the ISF.

 rriser@campus.haifa.ac.il