Time-delay matrix for chaotic cavities with non-ideal contacts

AURÉLIEN GRABSCH[1,3], DMITRY V. SAVIN[2] & CHRISTOPHE TEXIER[1]

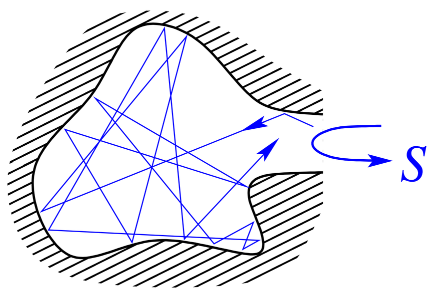
[1] Laboratoire de Physique Théorique et Modèles Statistiques, Université Paris-Sud, Orsay, France
[2] Department of Mathematics, Brunel University London, Uxbridge, UK ; [3] Leiden University, The Netherlands

SCATTERING IN PERFECTLY COUPLED CHAOTIC QDs

- **Wigner-Smith time-delay matrix :**

  $N\times N$ Scattering matrix $\mathcal{S}(\varepsilon)=\mathcal{U}\,e^{\mathrm{i}\Theta}\,\mathcal{U}^\dagger \quad\longrightarrow\quad \boxed{\mathcal{Q}\overset{\text{def}}{=}-\mathrm{i}\hbar\mathcal{S}^\dagger\partial_\varepsilon\mathcal{S}}$

  chaotic dynamics ⇒ Random Matrix Theory



- **Partial time delays :** e.v. $\{\tilde\tau_a\}$ of $\partial_\varepsilon\Theta$ (independent of basis)

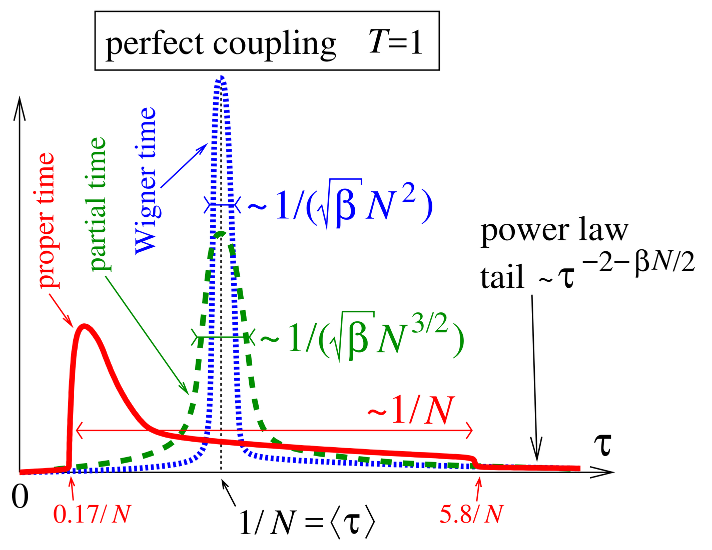
  marginal distrib. ⇒ Fyodorov, Savin & Sommers, J.Math.Phys. '97 ; PRE '97 ; PRE '01
- **Proper time delays :** e.v. $\{\tau_i=\tau_{\rm H}/\gamma_i\}$ of $\mathcal{Q}$ (with $\tau_{\rm H}=\frac{h}{\Delta}$) : Laguerre ensemble

  Brouwer, Frahm & Beenakker, PRL '97

  $$\mathcal{P}_N(\gamma_1,\cdots,\gamma_N)\propto\prod_{i<j}|\gamma_i-\gamma_j|^\beta\prod_{k=1}^N\gamma_k^{\beta N/2}\,e^{-\beta\gamma_k/2}$$
- **Wigner time delay :**

  trace $\tau_{\rm W}=(1/N)\,\mathrm{Tr}\,\{\mathcal{Q}\}=(1/N)\sum_a\tau_a=(1/N)\sum_a\tilde\tau_a$

  Distribution for $N\gg1$: Texier & Majumdar, PRL '13



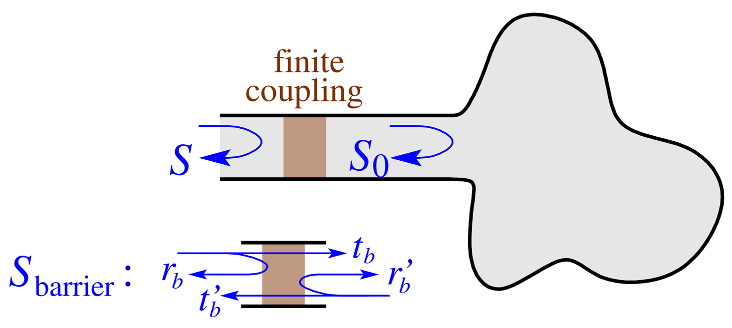
- RECENT REVIEW : Texier, Physica E **82**, 16 (2016) ; or arXiv:1507.00075

THE RANDOM MATRIX MODEL

- **Scattering matrix and reaction matrix :**

  $$\mathcal{S}=(\mathbf{1}_N-\mathrm{i}\mathcal{K})\,(\mathbf{1}_N+\mathrm{i}\mathcal{K})^{-1}\quad\text{with}\quad\mathcal{K}=\mathcal{W}^\dagger\,(\varepsilon-\mathcal{H}_0)^{-1}\,\mathcal{W}$$

  then $$\mathcal{Q}=-2\,(\mathbf{1}_N-\mathrm{i}\mathcal{K})^{-1}\,\frac{\partial\mathcal{K}}{\partial\varepsilon}\,(\mathbf{1}_N+\mathrm{i}\mathcal{K})^{-1}$$

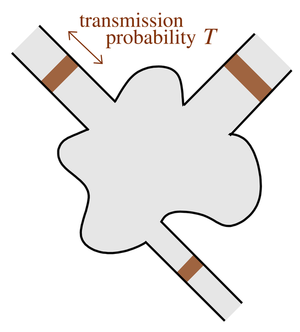


- **For arbitrary couplings :**

  perfect couplings : $\mathcal{K}_0\in$ Cauchy ensemble

  arbitrary couplings : $\mathcal{K}=\mathcal{C}\,\mathcal{K}_0\,\mathcal{C}$

  uniform couplings : $\mathcal{C}=\sqrt{\kappa}\,\mathbf{1}_N$ (transmission proba: $T=\frac{4\kappa}{(1+\kappa)^2}$)



WIGNER-SMITH MATRIX DISTRIBUTION FOR NON-IDEAL CONTACTS

$$\boxed{\mathcal{P}(\Gamma)\propto(\det\Gamma)^{\beta N/2}\int\mathrm{D}\mathcal{K}\,\frac{\det(\mathbf{1}_N+\mathcal{K}^2)^{\beta N/2}}{\det(\mathbf{1}_N+\kappa^2\mathcal{K}^2)^{1-\frac{\beta}{2}+\beta N}}\,\exp\left(-\frac{\beta}{2}\kappa\,\mathrm{Tr}\left\{\frac{\mathbf{1}_N+\mathcal{K}^2}{\mathbf{1}_N+\kappa^2\mathcal{K}^2}\Gamma\right\}\right)}\qquad(2)$$

→ for $\kappa=1$: recover Laguerre, $\mathcal{P}(\Gamma)\propto(\det\Gamma)^{\beta N/2}e^{-(\beta/2)\,\mathrm{Tr}\{\Gamma\}}$ (Brouwer, Frahm & Beenakker '97)

→ can be generalized to arbitrary couplings (when $\mathcal{C}$ is not $\propto\mathbf{1}_N$)

- Distribution of e.v. : use Harish-Chandra-Itzykson-Zuber integral

$$\mathcal{P}_N(\gamma_1,\cdots,\gamma_N)\propto\Delta_N(\gamma)\prod_n\gamma_n^N\int_{\mathbb{R}^N}\mathrm{d}k_1\cdots\mathrm{d}k_N\,\frac{\Delta_N(k)^2}{\Delta_N(k^2)}\prod_n\frac{(1+k_n^2)^N}{(1+\kappa^2k_n^2)^{N+1}}\det\left[\exp\left(-\kappa\,\frac{1+k_i^2}{1+\kappa^2k_i^2}\gamma_j\right)\right]\qquad(3)$$

WIGNER TIME DELAY DISTRIBUTION (UNIFORM COUPLINGS)

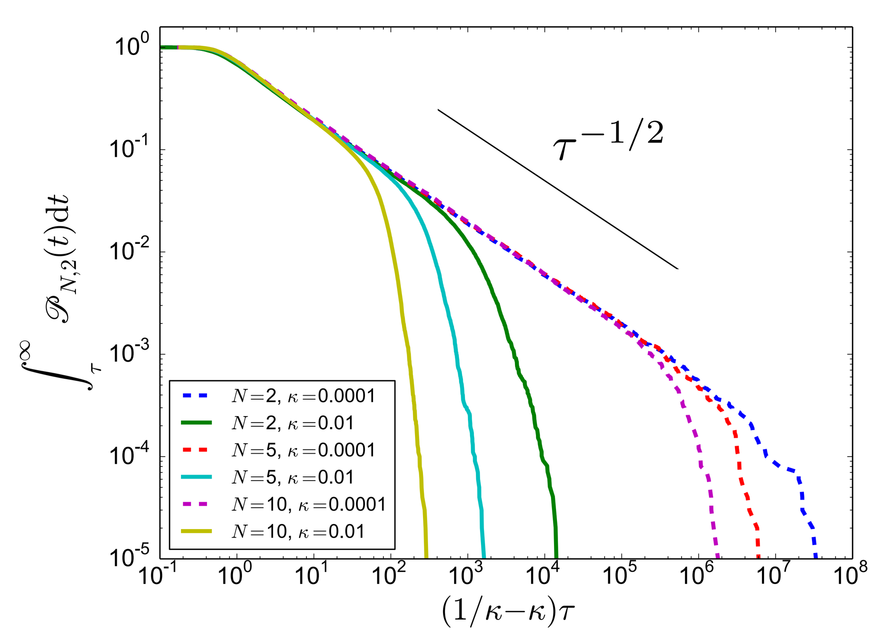
Wigner time delay distribution $\mathscr{P}_{N,\beta}(\tau)$ and generating function :

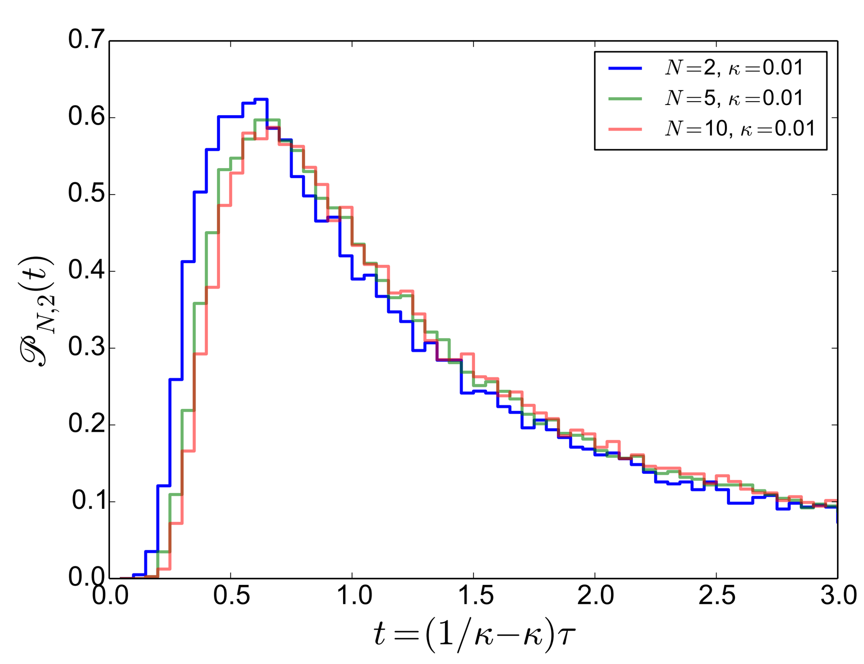
$$\mathcal{Z}_{N,\beta}(p)\propto\left\langle e^{-(2p/\beta\kappa)\,\mathrm{Tr}\{\Gamma^{-1}\}}\right\rangle\quad\leftrightarrow\quad\int_0^\infty\mathrm{d}\tau\,\mathscr{P}_{N,\beta}(\tau)\,e^{-(2Np/\beta\kappa)\tau}=\frac{\mathcal{Z}_{N,\beta}(p)}{\mathcal{Z}_{N,\beta}(0)}\qquad(4)$$

HCIZ + Andreief ⇒
$$\boxed{\mathcal{Z}_{N,2}(p)=\int_{\mathbb{R}^N}\mathrm{d}k_1\cdots\mathrm{d}k_N\,\frac{\Delta_N(k)^2}{\prod_n(1+k_n^2)^N}\,\frac{\det\left[\left(p\,\frac{1+\kappa^2k_j^2}{1+k_j^2}\right)^{\frac{N+i}{2}}K_{N+i}\left(2\sqrt{p\,\frac{1+k_j^2}{1+\kappa^2k_j^2}}\right)\right]}{\det\left[\left(\frac{1+k_j^2}{1+\kappa^2k_j^2}\right)^{-N-i}\right]}}\qquad(5)$$

Perfect couplings ($\kappa=1$) : $\mathcal{Z}_{N,2}(p)\propto\det\left[p^{\frac{N+i+j-1}{2}}\,K_{N+i+j-1}(2\sqrt{p})\right]$

NUMERICS



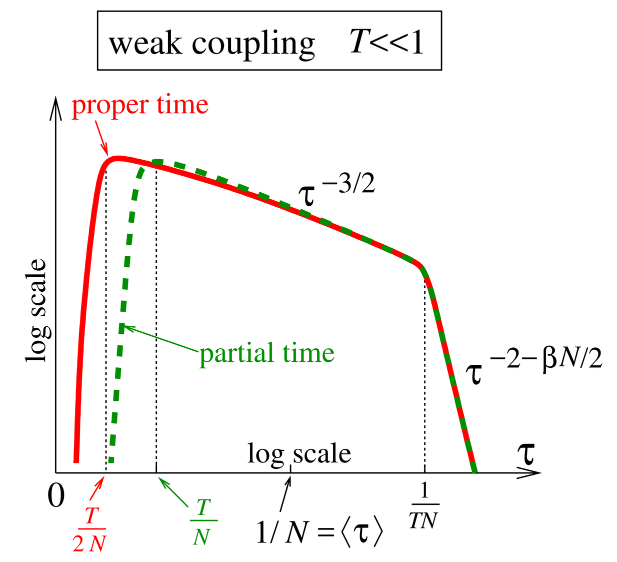


FINITE COUPLING

- Marginal of $\tilde\tau_a$'s : Fyodorov, Savin & Sommers '97
- In the limit $T\to0$, rescale $\tau=(\beta T/8)\,t$. One gets

$$\lim_{T\to0}\tilde q_{N,\beta}(t)=C_{N,\beta}\,\frac{e^{-1/t}}{t^{2+\beta N/2}}\,U\left(\frac12,\frac{\beta N+3}{2},\frac1t\right)$$

- Marginal of τ_a's : Sommers, Savin & Sokolov, PRL '01



- QUESTIONS :

  distribution of the matrix $\mathcal{Q}$? and $\tau_{\rm W}$?

FROM $\mathcal{Q}_{s0}$ TO $\mathcal{Q}_s$

$\mathcal{Q}_0=-2\kappa\,(\mathbf{1}_N-\mathrm{i}\,\kappa\,\mathcal{K}_0)^{-1}\,\partial_\varepsilon\mathcal{K}_0\,(\mathbf{1}_N+\mathrm{i}\,\kappa\,\mathcal{K}_0)^{-1}$, hence

$$\mathcal{Q}_s=A\,\mathcal{Q}_{s0}\,A\quad\text{with}\quad A=\sqrt{\kappa}\sqrt{\frac{\mathbf{1}_N+\mathcal{K}_0^2}{\mathbf{1}_N+\kappa^2\mathcal{K}_0^2}}\qquad(1)$$

- Cauchy $P_{\mathcal{K}}^{(0)}(\mathcal{K}_0)\propto\left[\det(\mathbf{1}_N+\mathcal{K}_0^2)\right]^{-1-\beta(N-1)/2}$
- Laguerre $P_\Gamma^{(0)}(\Gamma_0)\propto(\det\Gamma_0)^{\beta N/2}\;e^{-(\beta/2)\;\mathrm{Tr}\{\Gamma_0\}}$
- Wigner time delay matrix distribution

$$\mathcal{P}(\Gamma)=\left\langle\delta\left(\Gamma-A^{-1}\Gamma_0A^{-1}\right)\right\rangle_{\mathcal{K}_0,\Gamma_0}$$

($\mathcal{S}_0$ and $\mathcal{Q}_{s0}$ independent).

- use Jacobian :

$\Gamma_0=A^\dagger Y\,A\Rightarrow\mathrm{D}\Gamma_0=\mathrm{D}Y\left[\det(A^\dagger A)\right]^{1+\beta(N-1)/2}$

⇐

LIMIT $T\simeq4\kappa\to0$ THEN $p\to\infty$

$$\mathcal{Z}_{N,2}(p)\underset{p\gg1}{\simeq}\mathscr{A}_N\,p^{\frac{N^2}{2}}\,e^{-2N\sqrt{p}}$$

$\boxed{\mathscr{P}_{N,2}(\tau)\sim T^{-1}\,(T/\tau)^{N^2+3/2}\,e^{-NT/(4\tau)}}$ for $\tau\ll T$

→ can be generalised ∀ β

LIMIT $T\simeq4\kappa\to0$ THEN $p\to0$

$$\frac{\mathcal{Z}_{N,2}(p)}{\mathcal{Z}_{N,2}(0)}\underset{p\ll1}{\simeq}1-B_N\sqrt{p}\,.$$

$\boxed{\mathscr{P}_{N,2}(\tau)\sim T^{-1}\,(T/\tau)^{3/2}}$ for $T\ll\tau\ll\frac1T$

LIMIT $p\to0$ FIRST ($T\simeq4\kappa\ll1$)

One expects $\lfloor\beta N/2\rfloor$ finite moments $(\mathcal{Z}_{N,\beta}(p)/\mathcal{Z}_{N,\beta}(0)=1-\frac{2p}{\beta\kappa}+(\cdots)p^2+\cdots)$

$$\mathscr{P}_{N,\beta}(\tau)\underset{\tau\gg1/T}{\sim}T^2\,(T\tau)^{-2-\beta N/2}\quad\text{for }\tau\gg\frac1T$$

REFERENCE

A. Grabsch, D. V. Savin & C. Texier, *Wigner-Smith time-delay matrix in chaotic cavities with non-ideal contacts*, J. Phys. A: Math. Theor. **51**, 404001 (2018)