Recent Developments in Pseudo-Hermitian Hamiltonians

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Introduction

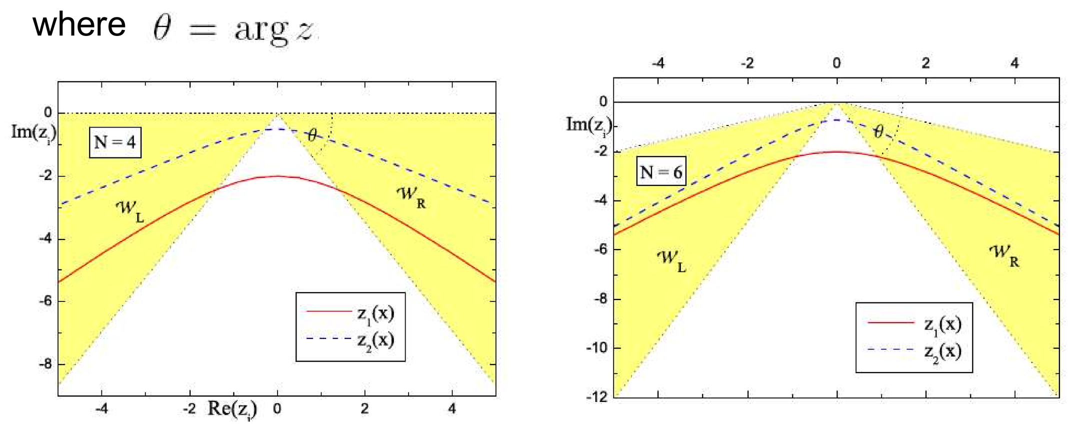
The interest in Non-Hermitian Hamiltonians was revived by the numerical observation [1] that the it is possible to obtain real spectra for the Hamiltonian

$$H = p^2 - g\,(\imath z)^N \qquad N \in \mathbb{N}$$ on a meaningful domain.

$\Phi(z) \to 0$, $z \to \infty$ } **Stokes lines**

$$\mathcal{W}_L(N) = \left\{ \theta \,\middle|\, -\frac{8+N}{2(N+2)}\pi < \theta < -\frac{4+N}{2(N+2)}\pi \right\},$$
$$\mathcal{W}_R(N) = \left\{ \theta \,\middle|\, -\frac{N}{2(N+2)}\pi < \theta < \frac{4-N}{2(N+2)}\pi \right\},$$

where $\theta = \arg z$



Possible parameterizations:
$$z_1(x) = x\cos(\theta_R^{AS}) + i\sin(\theta_R^{AS})\sqrt{a^2+x^2}$$
$$z_2(x) = -2i\sqrt{1+ix}$$

PT Symmetry $\quad [H, \mathcal{PT}] = 0$

Invariance under simultaneous **parity** and **time-reversal** transformations:

$$\mathcal{PT}: \quad p \longrightarrow p \qquad z \longrightarrow -z \qquad \imath \longrightarrow -\imath$$

Anti-unitary: $\mathcal{PT}\,(\alpha|\varphi\rangle + \beta|\psi\rangle) = \alpha^* \mathcal{PT}|\varphi\rangle + \beta^* \mathcal{PT}|\psi\rangle$

- **Unbroken PT-Symmetry:** $\mathcal{PT}|\Phi\rangle = |\Phi\rangle$ ⇓ **real eigenenergies**

$$\varepsilon|\Phi\rangle = H|\Phi\rangle = H\mathcal{PT}|\Phi\rangle = \mathcal{PT}H|\Phi\rangle\mathcal{PT}\varepsilon|\Phi\rangle = \varepsilon^*|\Phi\rangle$$

- **Broken PT-Symmetry:** $\mathcal{PT}|\Phi\rangle \neq |\Phi\rangle$ ⇓ **complex conjugate pairs**

$$H\mathcal{PT}|\Phi\rangle = \varepsilon^*\mathcal{PT}|\Phi\rangle$$

PT symmetry is a **sufficient** [2] but **not necessary** condition for the reality of the spectrum.

Hermitian ⟹ **Real eigenvalues**

⇕ **Pseudo-Hermiticity**

Complex eigenvalues ⟹ **Non-Hermitian**

Pseudo-Hermiticity

Similarity transformation: $\quad h = \eta H \eta^{-1} \quad \begin{cases} H^\dagger \neq H \\ h^\dagger = h \\ \eta^\dagger = \eta \end{cases}$

$H|\Phi\rangle = \varepsilon|\Phi\rangle$, $h|\phi\rangle = \varepsilon|\phi\rangle$ } Isospectral partners $\qquad |\phi\rangle = \eta|\Phi\rangle$

Bi-orthogonal basis $\quad \begin{cases} \langle\Phi_m|\eta^2|\Phi_n\rangle = \delta_{mn} \\ \sum_n \eta^2|\Phi_n\rangle\langle\Phi_n| = \mathbb{I} \end{cases}$

The Hamiltonian **H** is Hermitian with respect to a new metric [3]:

$$\langle\Phi|\Psi\rangle_\eta \equiv \langle\Phi|\eta^2\Psi\rangle$$

$$\langle\Phi|H\Psi\rangle_\eta = \langle\Phi|\eta^2 H\Psi\rangle = \langle\Phi|\eta h\eta\Psi\rangle = \langle\phi|h\psi\rangle =$$
$$= \langle h\phi|\psi\rangle = \langle\eta H\eta^{-1}\phi|\psi\rangle = \langle\eta^2 H\Phi|\Psi\rangle = \langle H\Phi|\Psi\rangle_\eta$$

The metric can be constructed from $\quad H^\dagger = \eta^2 H \eta^{-2}$

Symmetry operator $\quad \begin{cases} [H, S] = 0 \\ S \equiv \eta_a^{-2}\eta_b^2 \end{cases}$

⟹ Ambiguity in the construction of the metric.

Metrics for a general cubic Hamiltonian

The most general **polynomial** Non-Hermitian Hamiltonian maximally **cubic** in the position and momentum quantum operators, **x** and **p,** can be expressed as

$$H_C = \alpha_1\hat{p}^3 + \alpha_2\hat{p}^2 + \alpha_3\frac{\{\hat{p},\hat{x}^2\}}{2} + \alpha_4\hat{p} + \alpha_5\hat{x}^2 + \alpha_6 +$$
$$+\imath g\left(\alpha_7\frac{\{\hat{p}^2,\hat{x}\}}{2} + \alpha_8\frac{\{\hat{p},\hat{x}\}}{2} + \alpha_9\hat{x}^3 + \alpha_{10}\hat{x}\right)$$

and can be reduced to some very interesting cases, such as the **Swanson** Hamiltonian or the **Reggeon** single-site lattice model (expressed in terms of creation and annihilation operators) [4].

Isomorphism: $\quad F(\hat{x},\hat{p})G(\hat{x},\hat{p}) \cong F(x,p) \star G(x,p)$

operator functions in $\hat{x}$ and $\hat{p}$ — **scalar** functions multiplied according to **Moyal products**

$$F(x,p)\star G(x,p) \equiv F(x,p)\,e^{\frac{\imath}{2}\left(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x\right)}G(x,p)$$

$$F(\hat{x},\hat{p}) = \int_{-\infty}^{\infty} ds\,dt\, f(s,t)\,e^{i(s\hat{x}+t\hat{p})} \qquad F(x,p) = \int_{-\infty}^{\infty} ds\,dt\, f(s,t)\,e^{i(sx+tp)}$$

$$F(\hat{x},\hat{p})G(\hat{x},\hat{p}) = \int_{-\infty}^{\infty} ds\,dt\,ds'\,dt'\, f(s,t)f(s',t')\,e^{\frac{i}{2}(ts'-t's)}\,e^{i(s+s')\hat{x}+i(t+t')\hat{p}}$$

The Hamiltonian may be re-expressed using the correspondence:

$$p^n x^m = x^m p^n = \frac{m!n!}{(m+n)!}\sum_\pi x^m \star p^n \cong \frac{m!n!}{(m+n)!}\sum_\pi \hat{x}^m\hat{p}^n$$

⟹ differential equation for the function $\eta^2(x,p)$

$$H^\dagger(x,p)\star\eta^2(x,p) = \eta^2(x,p)\star H(x,p)$$

The Hermitian counterpart can be constructed from

$$h(x,p) = \eta(x,p)\star H(x,p)\star\eta^{-1}(x,p)$$

Exact Solutions:

$$H_c(x,p) = \alpha_3 px^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\alpha_9 x^3 + \frac{\alpha_4\alpha_9}{\alpha_3}x)$$
$$h_c(x,p) = \alpha_3 px^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 \qquad \eta(x,p) = e^{-g\frac{\alpha_9}{2\alpha_3}x^2}$$

$$H_c(x,p) = \alpha_2 p^2 + \alpha_3 px^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\frac{2\alpha_2\alpha_9}{\alpha_3}px + \alpha_9 x^3 + \frac{\alpha_4\alpha_9}{\alpha_3}x)$$
$$h_c(x,p) = \alpha_2 p^2 + \alpha_3 px^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2\frac{\alpha_2\,\alpha_9^2}{\alpha_3{}^2}x^2$$
$$\eta^2(x,p) = e^{-g\frac{\alpha_9}{\alpha_3}x^2}$$

$$H_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig(\alpha_7 p^2 x + \alpha_8 px + \alpha_{10}x)$$
$$h_c(x,p) = \alpha_1 p^3 + \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2\frac{(p^2\alpha_7 + p\alpha_8 + \alpha_{10})^2}{4\alpha_5}$$
$$\eta^2(x,p) = e^{g\left(\frac{\alpha_7}{3\alpha_5}p^3 + \frac{\alpha_8}{2\alpha_5}p^2 + \frac{\alpha_{10}}{\alpha_5}p\right)}$$

$$H_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + ig\,(\alpha_8 px + \alpha_{10}x)$$
$$h_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2\frac{\alpha_2\alpha_{10}^2}{\alpha_4^2}x^2 \qquad \text{for } \alpha_4 \neq 0,$$
$$h_c(x,p) = \alpha_2 p^2 + \alpha_4 p + \alpha_5 x^2 + \alpha_6 + g^2\frac{\alpha_8^2}{4\alpha_2}x^2 \qquad \text{for } \alpha_2 \neq 0.$$
$$\eta^2(x,p) = e^{-g\alpha_{10}/\alpha_4 x^2} \qquad \text{for } \alpha_4 \neq 0,$$
$$\eta^2(x,p) = e^{-g\alpha_8/2\alpha_2 x^2} \qquad \text{for } \alpha_2 \neq 0.$$

Symmetry Operator: for the Swanson Hamiltonian

$$H_S(x,p) = \alpha_2 p^2 + \alpha_5 x^2 + \imath g\alpha_8 px \quad \longleftrightarrow \quad S(\hat{x},\hat{p}) = e^{-g\frac{\alpha_8}{2\alpha_5}\hat{p}^2}e^{-g\frac{\alpha_8}{2\alpha_2}\hat{x}^2}$$

Perturbative Solutions:

$$H_{\rm SSLR}(x,p) = a^\dagger\star a + i\tilde{g}a^\dagger\star\left(a + a^\dagger\right)\star a = \frac{1}{2}(x^2+p^2-1) + ig(x^3 + p^2x - 2x)$$

$$(2x\partial_p - 2p\partial_x)\,\eta^2(x,p) = g\left(4x^3 - 8x + 4p^2x + 2p\partial_x\partial_p - 3x\partial_p^2 - x\partial_x^2\right)\eta^2(x,p)$$

$$\eta^2(x,p) = 2\sum_{n=0}^{\infty} g^n c_n(x,p) \qquad \eta(x,p) = 1 + \sum_{n=1}^{\infty} g^n q_n(x,p)$$

$$\begin{cases}
c_1(x,p) = p^3 - 2p + px^2, \\
c_2(x,p) = p^6 - 4p^4 + p^2 + x^2 - 4p^2x^2 + 2p^4x^2 + p^2x^4, \\
c_3(x,p) = \frac{2}{3}p^9 - 4p^7 - 5p^5 + 24p^3 - 4p + 8px^2 - 6p^3x^2 - 8p^5x^2 + 2p^7x^2 - px^4 + \frac{2}{3}p^3x^6 \\
\qquad -4p^3x^4 + 2p^5x^4, \\
c_4(x,p) = \frac{1}{3}p^{12} - \frac{8}{3}p^{10} - 12p^8 + 76p^6 - 5p^4 - 72p^2 + 24x^2 - 18p^2x^2 + 104p^4x^2 \\
\qquad -28p^6x^2 - 8p^8x^2 - 13x^4 + 28p^2x^4 - 20p^4x^4 - 8p^6x^4 + 2p^8x^4 - 4p^2x^6 \\
\qquad +\frac{4}{3}p^{10}x^2 - \frac{8}{3}p^4x^6 + \frac{4}{3}p^6x^6 + \frac{1}{3}p^4x^8.
\end{cases}$$

$$h_{\rm SSLR}(x,p) = \frac{1}{2}(x^2+p^2-1) + g^2\left(\frac{3}{2}p^4 - 4p^2 + 1 - 4x^2 + 3p^2x^2 + \frac{3}{2}x^4\right)$$
$$-g^4\left(\frac{17}{2}p^6 - 34p^4 + 4p^2 + 8 + 4x^2 - 48p^2x^2 + \frac{41}{2}p^4x^2 - 14x^4 + \frac{31}{2}p^2x^4 + \frac{7}{2}x^6\right) + \mathcal{O}(g^6)$$

Quantum Brachistochrone Problem

Passage time between two **orthogonal states**

- **Hermitian time evolution:**

$$\|\langle\phi_f|e^{-\imath\mathbf{h}\tau}|\phi_i\rangle\| = 1 \quad \Longleftrightarrow \quad \tau = \frac{\pi}{\omega_{fi}}$$

$\langle\phi_f|\phi_i\rangle = 0 \qquad$ Lower bound for fixed ω_{fi}

- **Non-Hermitian time evolution:**

$$H = \begin{pmatrix} re^{+\imath\theta} & s \\ s & re^{-\imath\theta} \end{pmatrix} \qquad h = \begin{pmatrix} r\cos\theta & -\frac{\omega}{2} \\ -\frac{\omega}{2} & r\cos\theta \end{pmatrix}$$

Normalized Eigenvectors ⟶

$$|\Phi_\pm\rangle_\alpha = \frac{e^{\frac{\imath\pi}{4}(1\mp1)}}{\sqrt{2\cos\alpha}}\begin{pmatrix} -e^{\pm\frac{\imath\alpha}{2}} \\ \mp e^{\mp\frac{\imath\alpha}{2}} \end{pmatrix} \qquad |\phi_\pm\rangle = \frac{e^{\frac{\imath\pi}{4}(1\pm1)}}{\sqrt{2}}\begin{pmatrix} 1 \\ \mp1 \end{pmatrix}$$

Similarity Transformation: $\quad \eta_\alpha = \frac{1}{\sqrt{\cos\alpha}}\begin{pmatrix} \sin\frac{\alpha}{2} & -\imath\cos\frac{\alpha}{2} \\ -\imath\cos\frac{\alpha}{2} & r\sin\frac{\alpha}{2} \end{pmatrix}$

⟶ New Metric: $\eta_\alpha^2 \qquad \sin\alpha \equiv \frac{r}{s}\sin\theta$

$$\|\langle\Phi_f|e^{-\imath\mathbf{H}\tau}|\Phi_i\rangle_\eta\| = \|\langle\phi_f|e^{-\imath\mathbf{h}\tau}|\phi_i\rangle_\eta\| = 1 \quad \Longleftrightarrow \quad \tau = \frac{\pi}{\omega_{fi}}$$

$$\|\langle\phi_f|e^{-\imath\mathbf{H}\tau}|\phi_i\rangle_\eta\| = 1 \quad \Longleftrightarrow \quad \tau = \frac{\pi}{\omega_{fi}} + \frac{2\alpha}{\omega_{fi}}$$

Tuneable passage time [5]

- **Non-Hermitian dissipative systems [6]:**

Real characteristic frequency

$$\tilde{H} = \begin{pmatrix} E+\varepsilon & 0 \\ 0 & E-\varepsilon \end{pmatrix} - \imath\lambda\begin{pmatrix} re^{+\imath\theta} & s \\ s & re^{-\imath\theta} \end{pmatrix}$$

$$\tilde{\omega} = 2\sqrt{(\varepsilon + r\lambda\sin\theta)^2 - \lambda^2 s^2} \quad \longrightarrow \quad \tau = \frac{\pi}{\omega_{fi}}$$

The passage time is equal to the lower bound for Hermitian systems

Complex characteristic frequency

$$\hat{H} = \begin{pmatrix} E+\varepsilon & 0 \\ 0 & E-\varepsilon \end{pmatrix} - \imath\frac{\lambda}{2}\begin{pmatrix} 2\cos^2\vartheta & \sin2\vartheta \\ \sin2\vartheta & 2\sin^2\vartheta \end{pmatrix} \qquad E,\varepsilon \in \mathbb{R} \qquad \lambda,\vartheta \in \mathbb{C}$$

$$\hat{h} = \eta_{\hat{\alpha}}\hat{H}\eta_{\hat{\alpha}}^{-1} = \begin{pmatrix} E - \imath\frac{\lambda}{2} & -\imath\frac{\hat{\omega}}{2} \\ -\imath\frac{\hat{\omega}}{2} & E - \imath\frac{\lambda}{2} \end{pmatrix} \qquad e^{-\imath\hat{\alpha}} = e^{-\imath(\hat{\alpha}_r + \imath\hat{\alpha}_i)} \equiv \frac{-\imath\lambda\sin2\vartheta}{2\varepsilon + \hat{\omega} - \imath\lambda\cos2\vartheta}$$

$$\hat{\omega} = \hat{\omega}_r + \imath\hat{\omega}_i = \sqrt{4\epsilon^2 - \lambda^2 - 4\imath\epsilon\lambda\cos2\vartheta}$$

minimal passage time:

$$\|\langle\hat{\phi}_f|e^{-\imath\hat{\mathbf{h}}\tau}|\hat{\phi}_i\rangle_{\hat{\eta}}\|^2 = \frac{1}{2}e^{-\lambda_r t}\left[\cosh(\hat{\omega}_i t) - \cos(\hat{\omega}_r t)\right]$$

$$\|\langle\hat{\phi}_f|e^{-\imath\hat{\mathbf{H}}\tau}|\hat{\phi}_i\rangle_{\hat{\eta}}\|^2 = \frac{\cosh(\hat{\omega}_i t) - \cos(2\hat{\alpha}_r - \hat{\omega}_r t)}{2e^{\lambda_r t}\cos\hat{\alpha}\cos\hat{\alpha}^*}$$

$$\frac{\cosh(\hat{\omega}_r t) - \cos(2\hat{\alpha}_r - \hat{\omega}_r t)}{\cosh^2\hat{\alpha}_i(1 + \cosh(\pi\frac{\hat{\omega}_i}{\hat{\omega}_r}))} = e^{\lambda_r(t - \frac{\pi}{\hat{\omega}_r})}$$

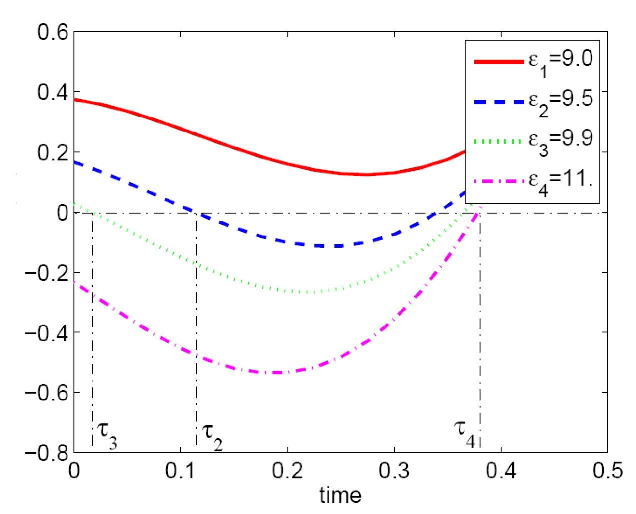


Figure: Difference between the Right- and Left-hand sides of the equation above for fixed values of $\vartheta, E, \hat{\omega}$. The parameters ε and λ vary accordingly.

Final Remarks

Non-Hermitian Hamiltonians may allow one to explore:

- New models with real eigenvalues
- New phenomena that do exist in Hermitian quantum physics.

References

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