The birth of a cut in unitary random matrix ensembles

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Brunel workshop on random matrix theory

December 18, 2007

Outline

1. Unitary random matrix ensembles
2. Critical ensembles
3. The birth of a cut
4. Riemann-Hilbert problems

1. Unitary random matrix ensembles

- Consider the set of Hermitian $n \times n$ matrices
- Probability measure of the form

$$\frac{1}{Z_n} \exp(-n \text{Tr}\, V(M))\, dM,$$

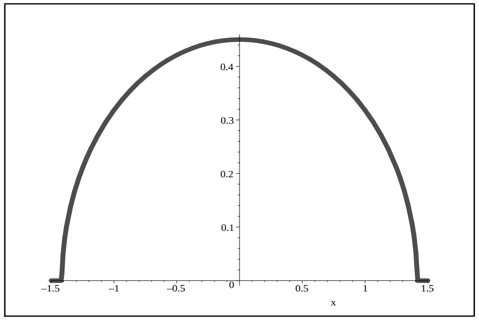
with

- V a real analytic function (e.g. a polynomial),
- V has enough growth at $\pm\infty$,
- $dM = \prod_{j=1}^{n} dM_{jj} \prod_{i<j} d\text{Re}\, M_{ij} d\text{Im}\, M_{ij}$
- Z_n is a normalization constant

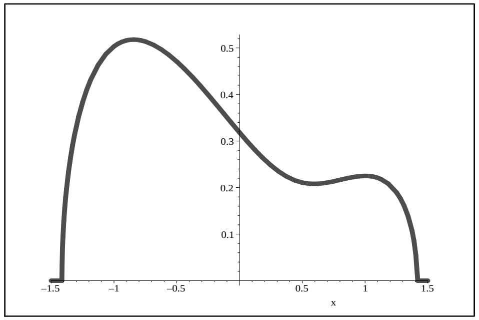
1. Unitary random matrix ensembles

Some well-known facts (1):

- limiting mean eigenvalue density depends on V
  - for $V(x) = x^2$, GUE, Wigner semi-circle law



  - for V of higher degree, density looks different, possibly more than one interval in the support



1. Unitary random matrix ensembles

Some well-known facts (2):

- Eigenvalues of a random matrix follow a determinantal point process with correlation kernel

$$K_n(x,y) = e^{-\frac{n}{2}V(x)} e^{-\frac{n}{2}V(y)} \sum_{k=0}^{n-1} p_k(x) p_k(y),$$

  p_k orthonormal polynomials w.r.t. weight $e^{-nV(x)}$ on $\mathbb{R}$

- Kernel contains information about eigenvalues
  - m-point correlation functions,
  - largest eigenvalue distribution,
  - gap probabilities, ...

1. Unitary random matrix ensembles

- We are interested in local behavior of eigenvalues near some reference point x^*
  - local scaling limits of the kernel

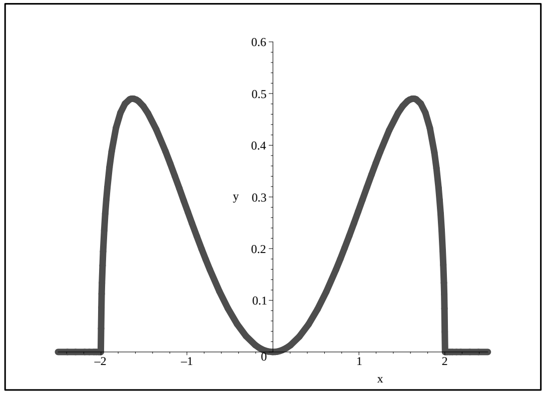
$$\lim_{n\to\infty} \frac{1}{cn^\beta} K_n(x^* + \frac{u}{cn^\beta}, x^* + \frac{v}{cn^\beta}) = ???$$

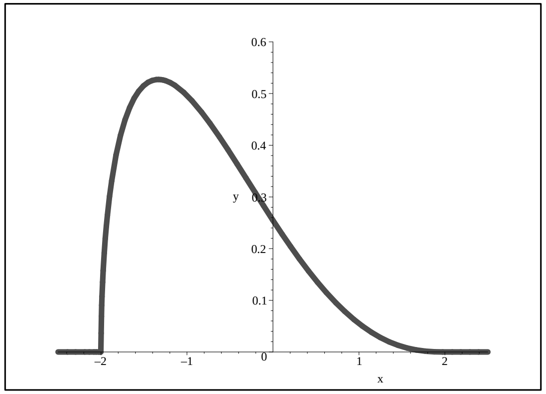
- in the bulk of the spectrum: sine kernel for $\beta = 1$, (Dyson, Deift-Kriechterbauer-McLaughlin-Venakides-Zhou, Bleher-Its, Pastur-Shcherbina)
- at the edge of the spectrum (generically): Airy kernel for $\beta = 2/3$ (Forrester, Tracy-Widom, DKMVZ, BI, Deift-Gioev)

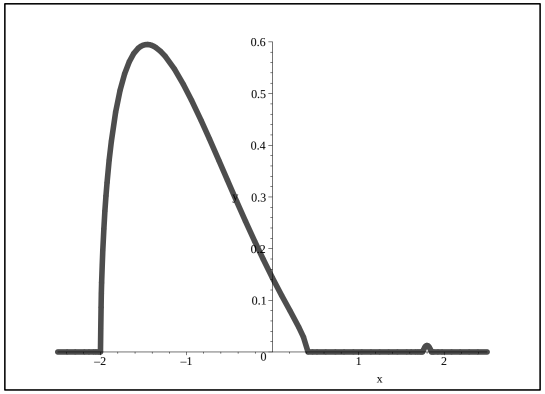
$\longrightarrow$ **Universality**

2. Critical ensembles

- Universality breaks down in three cases:



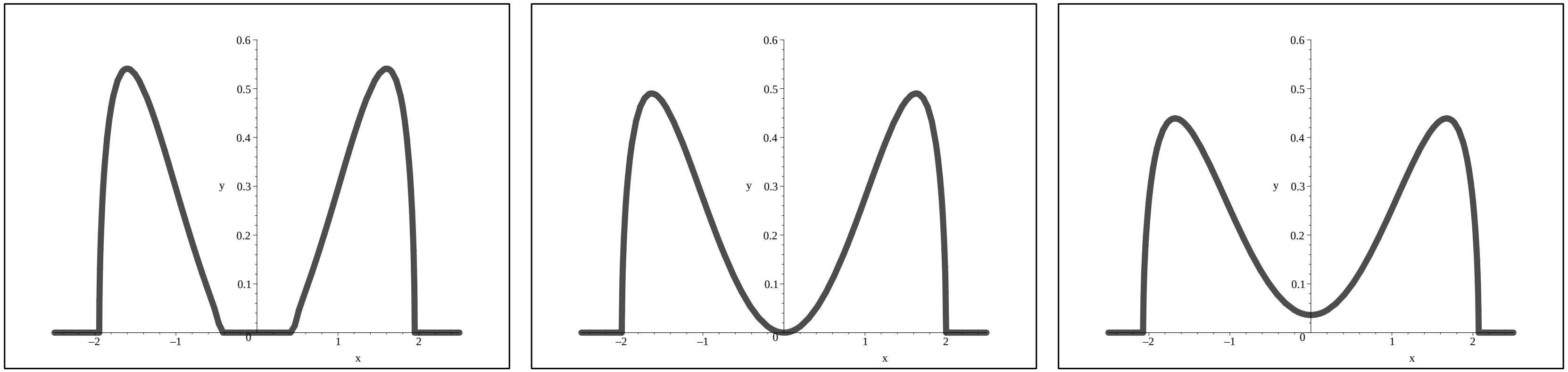




$\longrightarrow$ singular interior point, singular edge point, singular exterior point

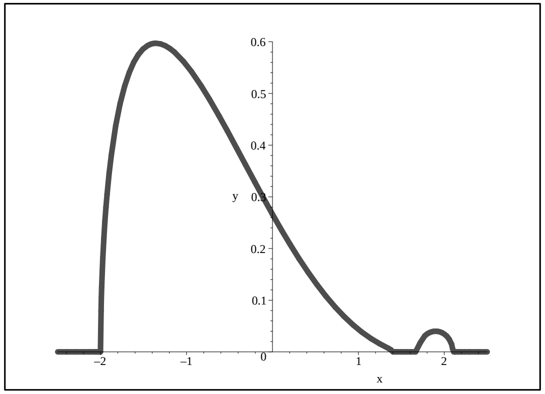
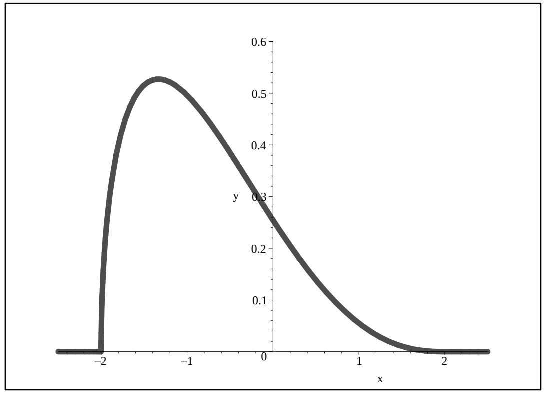
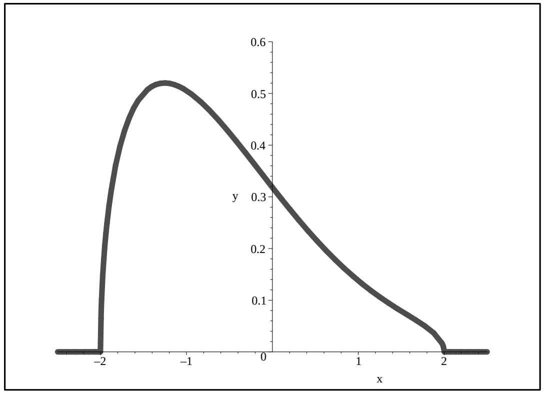
2. Critical ensembles

- Critical ensembles indicate a possible change of the number of intervals in the support
  - two merging intervals - opening of a gap



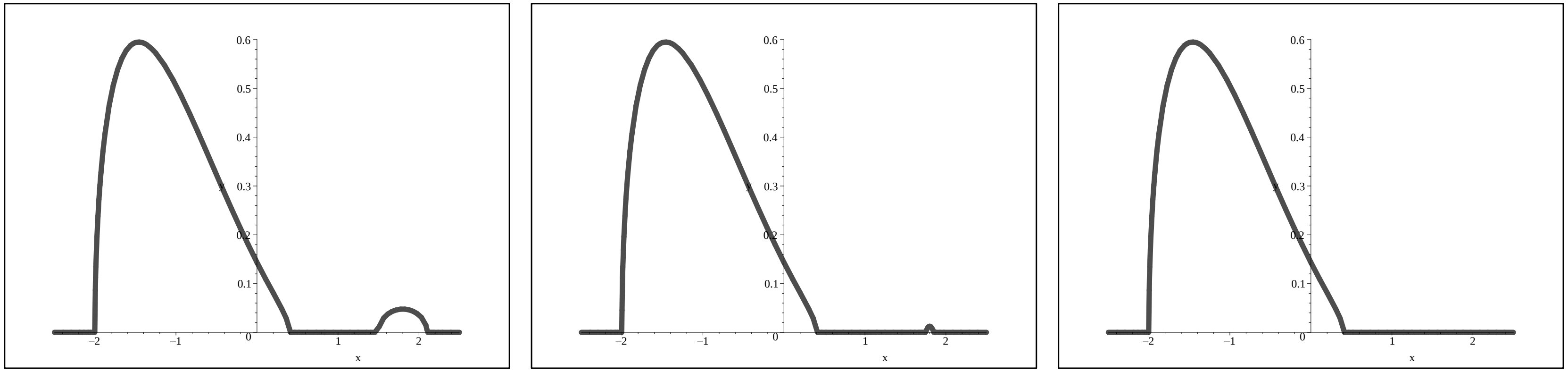
2. Critical ensembles

- Critical ensembles indicate a possible change of the number of intervals in the support
  - two merging intervals, with one of them shrinking at the same time

2. Critical ensembles

- Critical ensembles indicate a possible change of the number of intervals in the support
  - birth of a cut away from the spectrum - disappearing of an interval



2. Critical ensembles

- Including a parameter in the potential, $V = V_t$, leads to those transitions
- Local eigenvalue behavior in the transitions is described by double scaling limits of the eigenvalue correlation kernel
  - $n \to \infty$ and $t \to t_c$ at an appropriate rate
- no sine or Airy kernel in critical cases,
  - singular interior point: kernel built out of Ψ-functions for Painlevé II (Bleher-Its, TC-Kuijlaars, Shcherbina)
  - singular edge point: kernel built out of Ψ-functions for higher order Painlevé I (Bowick-Brézin, TC-Vanlessen)

3. The birth of a cut

The birth of a cut

- studied in physics literature by Eynard (2006)
- mathematics literature:
  ’independent and simultaneous’ works by
  Mo, Bertola-Lee, and myself
  (3 papers appeared on arXiv 19-26 november 2007)

The birth of a cut

- How does one recognize a critical potential V?
  - limiting mean eigenvalue density μ_V is an equilibrium measure minimizing

$$I_V(\mu) = \iint \log \frac{1}{|x-y|} d\mu(x) d\mu(y) + \int V(y) d\mu(y)$$

  - variational conditions

$$2\int \log|x-u| d\mu_V(u) - V(x) = \ell, \qquad \text{for } x \in \operatorname{supp} \mu_V,$$
$$2\int \log|x-u| d\mu_V(u) - V(x) \leq \ell, \qquad \text{for } x \in \mathbb{R} \setminus \operatorname{supp} \mu_V.$$

- singular exterior point occurs if there is equality outside the support

3. The birth of a cut

- We assume a potential V such that supp $\rho_V = [a, b]$, with a singular exterior point $x^* > b$,
- Additional assumption: variational equality at x^* is of order two (simplest case)
- We put $V_t = V/t$
  - $t < 1$: one interval
  - $t = 1$: singular point
  - $t > 1$: new cut is born

Those conditions can be satisfied for polynomials of degree 4 or higher.

3. The birth of a cut

- limiting kernel in the birth of a cut-transition?
- double scaling limit where we let $n \to \infty$ and at the same time $t \to 1$
  - appropriate rate of convergence turns out to be such that $t - 1 = \mathcal{O}\left(\frac{\log n}{n}\right)$
  - bounded number of eigenvalues expected in the new cut
- We write

$$\nu = c_V (t-1) \frac{n}{\log n}$$

and let $n \to \infty$, $t \to 1$ in such a way that

$$\nu \to \nu_0.$$

3. The birth of a cut

The result:

- In the double scaling limit, we have

$$\lim \frac{1}{(cn)^{1/2}} K_{n,t}\left(x^* + \frac{u}{(cn)^{1/2}}, x^* + \frac{v}{(cn)^{1/2}}\right)$$

$$= \begin{cases} \mathbb{K}^{\mathrm{GUE}}(u,v;k) & \text{for } k - \frac{1}{2} < \nu_0 < k + \frac{1}{2},\ k \geq 1, \\ 0 & \text{for } \nu_0 < 1/2, \end{cases}$$

$$\mathbb{K}^{\mathrm{GUE}}(u,v;k) = \frac{e^{-\frac{u^2+v^2}{2}}}{2^k \sqrt{\pi}(k-1)!} \frac{H_k(u)H_{k-1}(v) - H_k(v)H_{k-1}(u)}{u-v},$$

where H_k are Hermite polynomials

3. The birth of a cut

- $t < 1$, $\nu_0 < 0$: no eigenvalues expected, trivial limiting kernel
- when ν_0 increases, more eigenvalues 'move' to the new cut
- eigenvalues in the new cut seem to behave like the eigenvalues in a finite GUE
- Discontinuity of limiting kernel when ν_0 is a half integer?

3. The birth of a cut

$$\frac{1}{(cn)^{1/2}} K_{n,t}\left(x^* + \frac{u}{(cn)^{1/2}}, x^* + \frac{v}{(cn)^{1/2}}\right)$$

$$= \begin{cases} \lambda^-_{n,t}\mathbb{K}^{\mathrm{GUE}}(u,v;k) & \\ \qquad +\lambda^+_{n,t}\mathbb{K}^{\mathrm{GUE}}(u,v;k+1) + \mathcal{O}\left(\frac{\log n}{n^{1/2}}\right), & \text{for } k \leq \nu \leq k+1, \\ \mathcal{O}(n^{-1/2}), & \text{for } s < 0. \end{cases}$$

Furthermore the sequences $\lambda^{\pm}_{n,t}$ are such that

$$\lambda^+_{n,t} + \lambda^-_{n,t} = 1,$$

$$\lambda^+_{n,t} = 1 - \lambda^-_{n,t} = \mathcal{O}(n^{-1/2+\nu-k}), \qquad \text{as } k \leq \nu \leq k+1/2,$$

$$\lambda^-_{n,t} = 1 - \lambda^+_{n,t} = \mathcal{O}(n^{1/2+k-\nu}), \qquad \text{as } k+1/2 \leq \nu \leq k+1.$$

4. Riemann-Hilbert problems

- Proof based on the Deift/Zhou steepest descent analysis for the Riemann-Hilbert (RH) problem for orthogonal polynomials

(a) $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic.

(b) For $x \in \mathbb{R}$,

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-nV_t(x)} \\ 0 & 1 \end{pmatrix}, \qquad \text{for } x \in \mathbb{R}.$$

(c) Y has the following asymptotic behavior at infinity,

$$Y(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \qquad \text{as } z \to \infty.$$

4. Riemann-Hilbert problems

- Unique solution

$$Y(z) = \begin{pmatrix} \kappa_n^{-1} p_n(z) & \frac{\kappa_n^{-1}}{2\pi i} \int_{\mathbb{R}} \frac{p_n(u) e^{-nV_t(u)}}{u-z} du \\ -2\pi i \kappa_{n-1} p_{n-1}(z) & -\kappa_{n-1} \int_{\mathbb{R}} \frac{p_{n-1}(u) e^{-nV_t(u)}}{u-z} du \end{pmatrix}$$

in terms of OP's with respect to weight $e^{-nV_t(x)}$,

- Kernel can be expressed in terms of Y,

$$K_{n,t}(x,y) = e^{-\frac{n}{2}V_t(x)} e^{-\frac{n}{2}V_t(y)} \frac{1}{2\pi i(x-y)} \begin{pmatrix} 0 & 1 \end{pmatrix} Y_{\pm}^{-1}(y) Y_{\pm}(x) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4. Riemann-Hilbert problems

- Goal is to find asymptotics for Y in the double scaling limit
- Deift/Zhou steepest descent method
  - series of transformations, 'undressing' of the RH problem
  - $Y \mapsto T \mapsto S \mapsto R$
  - $R(z) = I + o(1) \quad \Rightarrow$ asymptotics for Y
  - Two crucial features
    - → Construction of g-function using modified equilibrium measures
    - → Construction of local parametrix near x^* using RH problem for Hermite polynomials

Comparison with Mo and Bertola-Lee

- Mo
  - he considers also higher order variational inequality
  - he does not consider transition near half integers
- Bertola-Lee
  - they consider also higher order variational inequality and transition near half integers
  - they do not construct the g-function explicitly and they restrict to a local deformation of the potential