Extreme Value Statistics of Maximal Eigenvalues of Random Matrices

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Plan of Talk

- Motivation - statistics and string theory
- Coulomb gas formulation for Gaussian ensembles
- Solution of problem by hard wall constraint- half Hilbert transfrom
- Some numerical tests

Conditioned matrix ensembles

M -random matrix e.g. from GOE, GUE or GSE ensemble

Say M describes Hamiltonian of vibrations about some stationary point in the energy landscape or the stability matrix of a fixed point in a dynamical system. Symmetry and complexity determine Gaussian statistics but in an experiment we can only observe systems where all eigenvalues are > z

(i) What is the probability of finding such a matrix ?
(ii) What are its spectral properties ?

Anthropic principles

The current form of string theory is believed to have 10^{500} possible vacua -the Landscape-each corresponding to a possible Universe and physical constants.

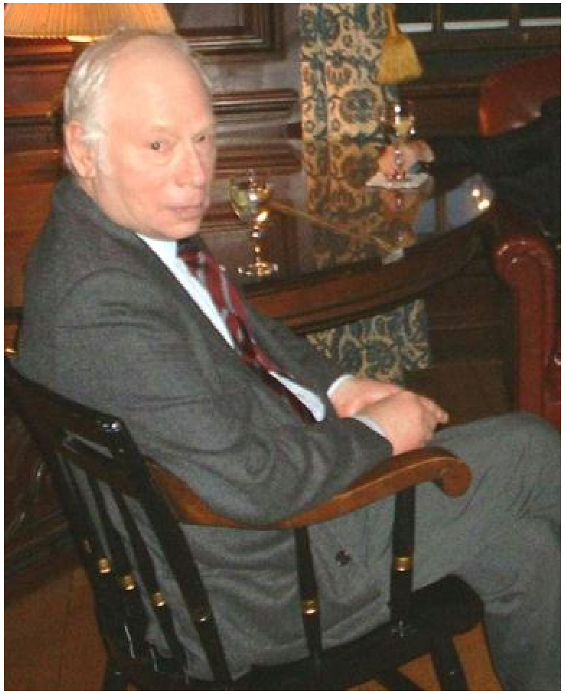
It seems unlikely that life can evolve in a generic universe so we are lucky to be here - anthropic principle this is the only type of Universe we could prossibly see, there could be lots of others (statistically dominating) which are quite different to ours.

What properties of our universe depend of the details of string theory and which depend simply on the statistics of the complex/random landscape ?

Anthropic Reasoning

Fred Hoyle- a ^{12}C based life form reasoned there should be a mechanism that permits sufficient ^{12}C for life to exist,he then found the ^{4}He+^{8}Be resonnance of ^{12}C that is responsible

Stephen Weinberg: the cosmological constant is small but if one takes the maximal value avoiding a « Big Rip » this gives a surprisngly good estimate



Estimating the probability of a stable universe

Aazami and R. Easther, J. Cosmol.Astropart. Phys. JCAP03 013 (2006)- classical string landscape is large dimensional (N)

If M describes the Hessian of a critical point what is the probability that it is a minimum, i.e. that all the eigenvalues are positive if M is chosen from GOE :

$$P(\lambda_{\min} > 0) \approx \exp(-N^2 \theta(0))$$

Aazami and Easther found an approximation (GOE) that $\theta(0)=1/4$ and numerically $\theta(0)=0.3291$ diagonalizing up to N =7 matrices.

But Gaussian random potential model is quite different- AJ Bray and DD PRL 2007

□□Distribution of Maximal Eigenvalues

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Tracy Widom Law - maximal eigenvalue situated at Wigner semi-circle edge and narrowly distributed

Coulomb Gas Formulation

Joint pdf for eigenvalues of Gaussian matrices

$$P(\{\lambda_i\}) = B_N \exp\left(-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i^2\right)\prod_{i<j}|\lambda_i - \lambda_j|^{\beta}$$

β= 1	2	4
GOE	GUE	GSE

$$\beta = \frac{1}{k_B T}$$

Rescaling

$$\lambda_i = \sqrt{N}\,\mu_i$$

Hamiltionian

$$H = \frac{N}{2}\sum_i \mu_i^2 - \sum_{i<j}\ln\left(|\mu_i - \mu_j|\right)$$

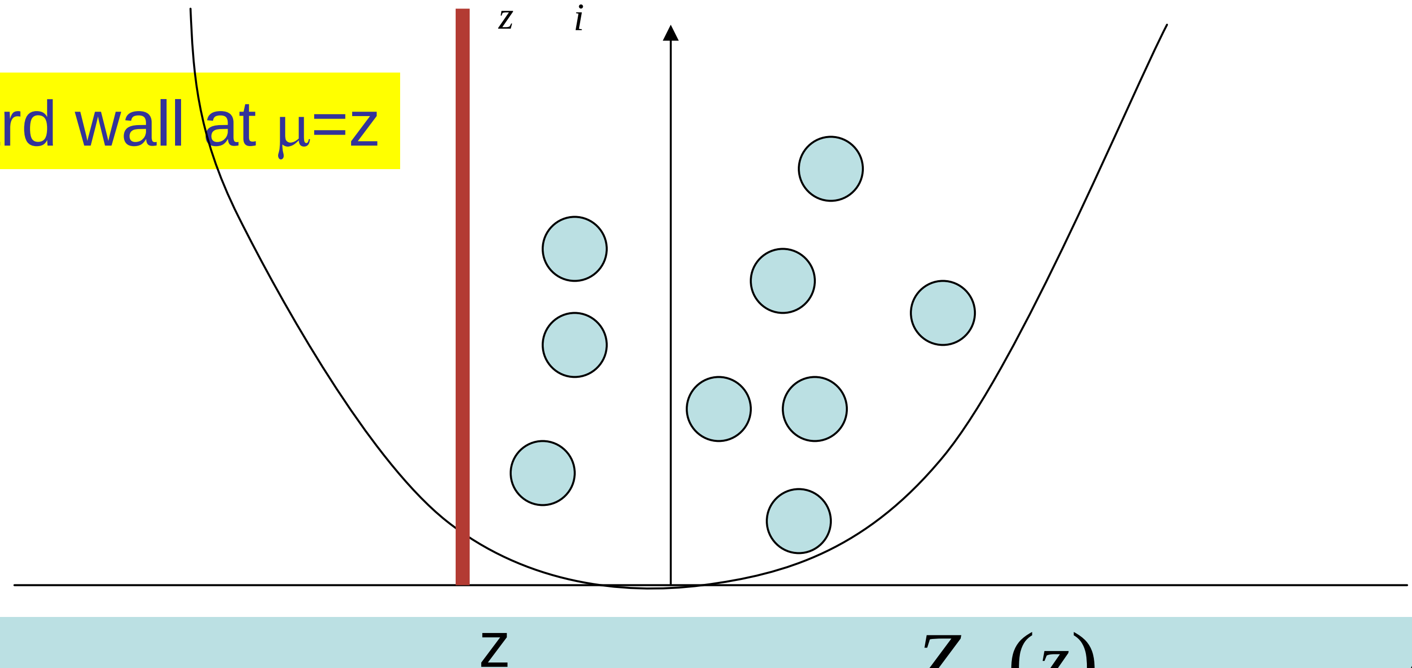
Canonical Partition function

$$Z_N = \int_{-\infty}^{\infty}\prod_i d\mu_i \exp(-\beta H)$$

Hard Wall Constraint

$$Z_N(z) = \int_z^\infty \prod_i d\mu_i \exp(-\beta H)$$

Put a hard wall at μ=z



$$P_N(\lambda_{\min} > \zeta) = P_N(\mu_{\min} > z) = \frac{Z_N(z)}{Z_N(-\infty)} \quad ; z = \frac{\zeta}{\sqrt{N}}$$

$$Z_N = Z_N(-\infty)$$

Density functional method

$$f(\mu) = \frac{1}{N}\sum_{i=1}^{N}\delta(\mu-\mu_i)$$

Normalised density field

$$H[f] = -N^2\Sigma[f]$$

Superextensive energy scaling

$$\Sigma[f] = -\frac{1}{2}\int f(\mu)\mu^2 d\mu + \frac{1}{2}\int f(\mu) f(\mu')\ln(|\mu-\mu'|)d\mu d\mu'$$

Constraints on f: $f(\mu) = 0 \;\; for \;\; \mu < z \quad ; \quad \int f(\mu)d\mu = 1$

$$Z_N(z) = \int d[f]J[f]\exp(N^2\beta\Sigma[f])$$

Jacobian to pass from coordinates
To density field

Functional Integral

$$J[f] = C_N \int \prod_i d\mu_i \delta(Nf(\mu) - \sum_i \delta(\mu - \mu_i))$$

$$= C_N \int d[g] \exp(N \int f(\mu) g(\mu) + N \ln\left(\int \exp(-g(\mu)) d\mu\right))$$

Jacobian term is exp(O(N)) and is thus negligible with respect to the energy term exp(O(N^2)).

Formally

$$J[f] = C_N^{''} \exp\left(-N \int d\mu f(\mu) \ln(f(\mu))\right)$$

If large N saddle-point can be justified above - mean field like entropy term (slightly more to it than that).

Leading order behaviour

Saddle point evaluation of functional integral

$$Z_N(z) = D_N \exp(N^2 \beta S(z) + O(N))$$

$$S[z] = \max_f \{\Sigma[f]\}$$

Saddle point equation, gives thermodynamically dominant density profile

$$\frac{\mu^2}{2} + C = \int_z^{\infty} d\mu' f(\mu') \ln(|\mu - \mu'|)$$

Differentiate w.r.t. μ (see Mehta) to get

$$\mu = P \int_z^{\infty} d\mu' \frac{f(\mu)}{\mu - \mu'}$$

Shift coordinates so x>0

$$\mu = z + x \Rightarrow z + x = (H_+ f)(x)$$

$$H_+ F = P \int_0^{\infty} dx' \frac{F(x')}{x - x'}$$

Half Hilbert Transform

Inversion of Half Hilbert Transform

Tricomi's theorem: for a function f with support on [0,L]

$$f(x) = (H_+^{-1}F)(x) = -\frac{1}{\pi^2\sqrt{x(L-x)}}\left\{P\int_0^L dx'\sqrt{x'(L-x')}\,\frac{F(x')}{x-x'} + C\right\}$$

C determined from: $f(L) = 0$

L determined from normalisation of f

Find

$$f(x) = \frac{1}{2\pi\sqrt{x}}\sqrt{L(z)-x}\,\big(L(z)+2x+2z\big); \quad x \in [0, L(z)]$$

Density field has support of length

$$L(z) = \frac{2}{3}\left[\sqrt{z^2+6} - z\right]$$

Recovering Wigner's Semi-Circle Law

However f must be real and positive, but f(0)<0 if

$$L(z) < -2z$$

$$\Rightarrow z < -\sqrt{2}$$

$$f(\mu) = \frac{1}{\pi}\sqrt{2-\mu^2}$$

Wigner semi-circle law when z=-√2- but this solution is Ok for all z<-√2 as it respects the boundary conditions

Setting the boundary to the left of z=-√2 does not affect the Coulomb gas - the wall does not touch the Wigner semi-cirlcle

$$\Rightarrow S(z) = S(-\sqrt{2}) = S(-\infty) \;\; for\; z < -\sqrt{2}$$

Modified density function

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

z=1/2

z=-1

z=0

Effect of moving the barrier position z on the eigenvalue density distribution

Comparsion with Numerics

QuickTime™ and a
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Analytic formula for
f for z=0 compared
to that found by
numerical diagonalizing
of 6x6 matrices

Computing S(z)

$$\frac{\mu^2}{2} + C = \int_z^{\infty} d\mu' f(\mu') \ln\left(|\mu - \mu'|\right) \Rightarrow$$

$$\int d\mu f(\mu) \frac{\mu^2}{4} + \frac{C}{2} = \frac{1}{2} \int d\mu d\mu' f(\mu) f(\mu') \ln(|\mu - \mu'|)$$

$$\Sigma[f] = \frac{1}{2} \int f(\mu)\mu^2 d\mu - \frac{1}{2} \int f(\mu) f(\mu') \ln(|\mu - \mu'|) d\mu d\mu'$$

$$S(z) = \frac{1}{4} \int f(\mu)\mu^2 d\mu - \frac{C}{2}$$

$$= \frac{1}{4} \int f(\mu)\mu^2 d\mu - \frac{1}{2} \int f(\mu) \ln(|\mu|) d\mu$$

Form of S(z)

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$$P_N(\lambda_{\min} > \zeta) = P_N(\mu_{\min} > z) = \frac{Z_N(z)}{Z_N(-\infty)} \quad ; z = \frac{\zeta}{\sqrt{N}}$$

$$\approx \exp(-N^2 \beta \theta(z)) ; \quad \theta(z) = S(-\sqrt{2}) - S(z)$$

$$\theta(-\sqrt{2} + y) = S(-\sqrt{2}) - S(-\sqrt{2} + y)$$

$$\approx \frac{y^3}{6\sqrt{2}}$$

Matching with Tracy Widom

For $y = \frac{t}{\sqrt{N}} + \sqrt{2} << 1 \quad but \quad O(1)$ i.e. t close to lower edge of Wigner semi-circle

$$P(\lambda_{\min} > t) \approx \exp(-N^2 \beta \theta(\frac{t}{\sqrt{N}} + \sqrt{2} - \sqrt{2})) \approx \exp(-N^2 \beta \frac{\left(\frac{t}{\sqrt{N}} + \sqrt{2}\right)^3}{6\sqrt{2}})$$

$$\approx \exp(-\frac{\beta}{24}\left(\sqrt{2}\, N^{\frac{1}{6}}(t + \sqrt{2N})\right)^3)$$

Matches with TW in same regime

Our result is valid within the Wigner sea but matches with TW close to -√2N- outside the sea $O(N^2)$ terms are no longer dominant and so need dnother method

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The persistence problem

Probability that all eigenvalues are positive $\approx \exp(-N^2 \beta \theta(0))$

$$\theta(0) = \frac{\ln(3)}{4} \approx 0.274653..$$

Note that $0.25 < 0.274653.. < 0.3291$

Aazami Easther aproximation

Exact result

Aazami Easther numerics

Numerical test of persistence

$$-0.272N^2 - 0.493N + 0.244$$

$\ln(P_N)$

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Direct enumeraton

Can generate the ensembles and do the diagonalization efficiently but for large N you never see matrices with all eigenvalues positive

Estimate from basic ensemble

$$P(\lambda_{\min} > 0) = \frac{m_+}{m}$$

positive

total #

Trick of limited usefulness: if M is a positve matrix then:

$$(v, Mv) > 0 \quad \forall v \Rightarrow (e_i, Me_i) = M_{ii} > 0$$

Generate new ensemble from old one where $M_{ii} \to |M_{ii}|$

Estimate from new ensemble

$$P(\lambda_{\min} > 0) = \frac{1}{2^N}\frac{m_+}{m}$$

positive

total #

Prob all M_{ii}>0

MonteCarlo

Direct enumeration is no longer practical for N>10

$$\frac{1}{(2\pi)^{\frac{N}{2}}} Z_N(-\infty) = \left\langle \prod_{i<j} |\lambda_i - \lambda_j| \right\rangle = g$$

Where < > is expectation over λ_i i.i.d with N(0,1) distribution

$$\frac{2^N}{(2\pi)^{\frac{N}{2}}} Z_N(0) = \left\langle \prod_{i<j} |\lambda_i - \lambda_j| \right\rangle_+ = g_+$$

Where $< >_+$ is expectation over λ_I i.i.d with |N(0,1)| distribution

$$\Rightarrow P_N(\lambda_{\min} > 0) = \frac{g_+}{2^N g}$$

Difficult to get accurate Results for very large N -big fluctuations in the Running average.

Conclusions

Coulomb gas formulation of eigenvalues of GOE GUE and GSE eigenvalues allows one to extract probability distribution of smallest eigenvalue far away from its typical value but within the sea

See a modified density of states in the condtioned ensemble quite rich behaviour

Is it possible to calculate the O(N) term and other lower order terms ?

Analysis of non symmetric matrix ensembles, Lyapounov exponents of dynamical fixed points of complex systems.

Are there more efficient ways of doing numerics ?

Can try to look at index distribution of random matrices