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Arbitrary Unitarily Invariant Random Matrix Ensembles and Supersymmetry

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Outline

- the problem and its history
- if you wish: a little bit about supersymmetry
- first step: supersymmetric representation for norm–dependent ensembles
- general case: supersymmetric representation for arbitrary rotation invariant ensembles
- some results beyond orthogonal polynomials

TG, J. Phys. A39 (2006) 12327, J. Phys. A39 (2006) 13191

The Problem and its History

Efetov's supersymmetry approach (early 80's) based on Gaussian assumption for probability densities.

physics: acceptable because of local universality

mathematics: fundamental restriction of supersymmetry ?

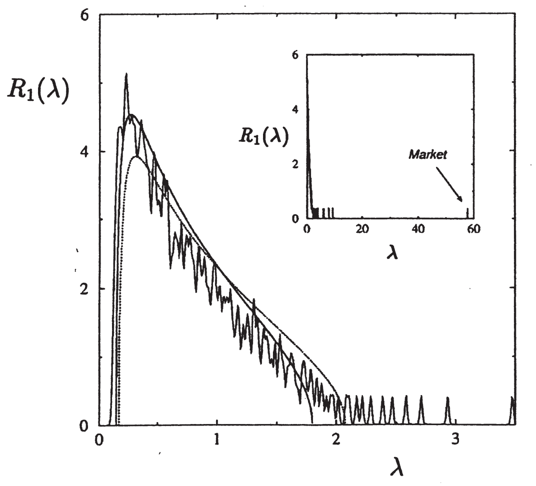
Hackenbroich, Weidenmüller (1995): universality proof involving supersymmetry and twofold asymptotics, not exact

Efetov, Schwiete, Takahashi (2004): superbosonization

TG (2006): algebraic duality, explicit construction

Littelmann, Sommers, Zirnbauer (2007): rigorous, threefold way

Need for Non–Gaussian Probability Densities



financial correlation matrices

empirical result deviate from Gaussian assumption

Laloux, Cizeau, Bouchaud, Potters (1999)

high–energy physics and quantum gravity, probability density:

$$P(H) \sim \exp\left(-\mathrm{tr}\, V(H)\right) \ , \qquad V(H) = \sum_j c_j H^j$$

large–scale universality, but $1/N$ expansion might be interesting

Mehta–Mahoux and Factorization

rotation–invariant probability density: $P(H) = P(E)$

factorization: $P(E) = \prod_{n=1}^{N} P^{(\mathrm{ev})}(E_n)$

$$R_k(E_1, \ldots, E_k) \;=\; \det\left[K_N(E_p, E_q)\right]_{p,q=1,\ldots,k}$$

$$K_N(E_p, E_q) \;=\; \sqrt{P^{(\mathrm{ev})}(E_p) P^{(\mathrm{ev})}(E_q)} \sum_{n=0}^{N-1} \omega_n(E_p)\omega_n(E_q)$$

$\omega_n(E_p)$ are orthogonal polynomials to the weight $P^{(\mathrm{ev})}(E_p)$

what if probability density does not factorize ??

Supersymmetry — Variables

k_1 complex commuting variables $\quad z_p,\ p = 1, \ldots, k_1$

k_2 complex anticommuting variables $\quad \zeta_p,\ p = 1, \ldots, k_2$

$\zeta_p\,\zeta_q \;=\; -\zeta_q\,\zeta_p$, in particular $\zeta_p^2 \;=\; 0$

every function is a finite polynomial, for example for $k_2 = 2$

$$f(\zeta_1, \zeta_2) \;=\; c_0 + c_{11}\zeta_1 + c_{12}\zeta_2 + c_2\zeta_1\zeta_2$$

complex conjugation $\quad \zeta_p \;\longrightarrow\; \zeta_p^* \;\longrightarrow\; \zeta_p^{**} = -\zeta_p$

$$\zeta_p\,\zeta_q^* \;=\; -\zeta_q^*\,\zeta_p$$

commuting and anticommuting variables commute

$z_p\,\zeta_q \;=\; \zeta_q\,z_p \quad$ and $\quad z_p\,\zeta_q^* \;=\; \zeta_q^*\,z_p$

Supersymmetry — Linear Algebra

supervectors $\psi = \begin{bmatrix} z \\ \zeta \end{bmatrix}$ and supermatrices $\sigma = \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix}$

matrices a, b have commuting entries
matrices μ, ν have anticommuting entries

$$\sigma\psi = \begin{bmatrix} a & \mu \\ \nu & b \end{bmatrix} \begin{bmatrix} z \\ \zeta \end{bmatrix} = \begin{bmatrix} az + \mu\zeta \\ \nu z + b\zeta \end{bmatrix} = \begin{bmatrix} z' \\ \zeta' \end{bmatrix} = \psi'$$

supertrace $\operatorname{trg}\sigma = \operatorname{tr} a - \operatorname{tr} b \quad \longrightarrow \quad \operatorname{trg}\sigma_1\sigma_2 = \operatorname{trg}\sigma_2\sigma_1$

superdeterminant $\operatorname{detg}\sigma = \dfrac{\det(a - \mu b^{-1}\nu)}{\det b}$

$$\longrightarrow \quad \operatorname{detg}\sigma_1\sigma_2 = \operatorname{detg}\sigma_1 \operatorname{detg}\sigma_2$$

Supersymmetry — Analysis

derivative $\quad \dfrac{\partial \zeta_p}{\partial \zeta_q} = \delta_{pq}$ and $\dfrac{\partial \zeta_p^*}{\partial \zeta_q} = 0$

Berezin integral $\quad \int d\zeta_p = 0$ and $\int \zeta_p d\zeta_p = \dfrac{1}{\sqrt{2\pi}}$

for example $\quad \int \exp(-a\zeta_p^*\zeta_p) d\zeta_p^* d\zeta_p = \int \left(1 - a\zeta_p^*\zeta_p\right) d\zeta_p^* d\zeta_p = \dfrac{a}{2\pi}$

apart from factors, derivative and integral are the same !

change of variables $\quad \psi \;\rightarrow\; \chi = \chi(\psi)$ requires

Jacobian or Berezinian $\quad \int f(\psi) d[\psi] = \int f(\psi(\chi)) \mathrm{detg} \dfrac{\partial \psi}{\partial \chi} d[\chi]$

Gaussian Integrals over Supervectors

matrix a has commuting entries

$$\int \exp(-z^\dagger a z) d[z] = \det{}^{-1} \frac{a}{2\pi} \quad \text{and} \quad \int \exp(-\zeta^\dagger a \zeta) d[\zeta] = \det \frac{a}{2\pi}$$

σ is a supermatrix

$$\int \exp(-\psi^\dagger \sigma \psi) d[\psi] = \operatorname{detg}^{-1} \frac{\sigma}{2\pi}$$

$\longrightarrow$ divergencies removed $\longrightarrow$ renormalization

$\longrightarrow$ Random Matrix Theory and disordered systems

Supersymmetry and Gaussian Random Matrices

Gaussian ensemble of $N \times N$ Hermitean random matrices H

k–point correlations $\quad R_k(x_1, \dots, x_k) = \left. \frac{\partial^k}{\prod_{p=1}^k \partial J_p} Z_k(x+J) \right|_{J=0}$

generating function obeys the identity (yes, this is exact!)

$$Z_k(x+J) = \int d[H] \exp(-\mathrm{tr}\, H^2) \prod_{p=1}^{k} \frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)}$$

$$= \int d[\sigma] \exp(-\mathrm{trg}\, \sigma^2)\, \mathrm{detg}^{-N}(\sigma - x - J)$$

where σ is a $2k \times 2k$ supermatrix

$\longrightarrow$ drastic reduction of dimensions

Efetov (1983)

Posing the Problem as a Structural Issue

can we generalize this to non–Gaussian probability densities ?

is there an identity of the form

$$\int d[H] P(H) \prod_{p=1}^{k} \frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)} = \int d[\sigma] Q(\sigma) \mathrm{detg}^{-N}(\sigma - x - J)$$

given an arbitrary rotation–invariant $P(H)$, what is $Q(\sigma)$?

First Step: Norm–dependent Ensembles

consider $P(H) = P^{(T)}(\mathrm{tr}\, H^2)$ $\longrightarrow$ transformation formula

$$Q^{(T)}(w) = \int_0^\infty P^{(T)}(u+w)u^{N^2/2-1}du \ , \qquad w = \mathrm{trg}\, \sigma^2$$

$\longrightarrow$ $Q(\sigma) = Q^{(T)}(\mathrm{trg}\, \sigma^2)$ also norm–dependent

inverse transformation: $P^{(T)}(u) = \dfrac{\partial^{N^2/2}}{\partial u^{N^2/2}} Q^{(T)}(u)$

dimensional reduction for norm–dependent ensembles

Examples

always $u = \mathrm{tr}\, H^2$ and $w = \mathrm{trg}\, \sigma^2$

fixed trace ensemble:

$$P^{(T)}(u) = a_0 \Theta\left(a_1 - u\right) \quad \longrightarrow \quad Q^{(T)}(w) = \frac{(a_1 - w)^{N^2/2}}{a_1^{N^2/2}} \Theta\left(a_1 - w\right)$$

non–extensive entropy ensemble:

$$P^{(T)}(u) = a_0 \left(1 + \frac{\kappa}{\Lambda} u\right)^{-1/(q-1)} , \quad 1 < q < 1 + \frac{2}{N^2} , \; \Lambda = \frac{1}{q-1} - \frac{N^2}{2}$$

$$\longrightarrow \quad Q^{(T)}(w) = \left(1 + \frac{\kappa}{\Lambda} w\right)^{-\Lambda}$$

Arbitrary Rotation–invariant Ensembles

use bosonic fields z_p and fermionic fields ζ_p

$$\frac{\det(H - x_p - J_p)}{\det(H - x_p + J_p)} = \int d[z_p] \exp(-iz_p^\dagger(H - x_p + J_p)z_p)$$

$$\int d[\zeta_p] \exp(-i\zeta_p^\dagger(H - x_p - J_p)\zeta_p)$$

characteristic function: $\Phi(K) = \int d[H]\, P(H) \exp\left(i\mathrm{tr}\, HK\right)$

Fourier matrix variable: $K = \sum_{p=1}^{k} z_p z_p^\dagger - \sum_{p=1}^{k} \zeta_p \zeta_p^\dagger$

$P(H)$ rotation invariant $\longrightarrow$ $\Phi(K)$ rotation invariant

Duality between Ordinary and Superspace

introduce $N \times 2k$ supermatrix $\quad A = [z_1 \cdots z_k \; \zeta_1 \cdots \zeta_k]$

$$K = \sum_{p=1}^{k} z_p z_p^\dagger - \sum_{p=1}^{k} \zeta_p \zeta_p^\dagger = AA^\dagger$$

$$B = A^\dagger A = \begin{bmatrix} z_1^\dagger z_1 & \cdots & z_1^\dagger z_k & z_1^\dagger \zeta_1 & \cdots & z_1^\dagger \zeta_k \\ \vdots & & \vdots & \vdots & & \vdots \\ z_k^\dagger z_1 & \cdots & z_k^\dagger z_k & z_k^\dagger \zeta_1 & \cdots & z_k^\dagger \zeta_k \\ -\zeta_1^\dagger z_1 & \cdots & -\zeta_1^\dagger z_k & -\zeta_1^\dagger \zeta_1 & \cdots & -\zeta_1^\dagger \zeta_k \\ \vdots & & \vdots & \vdots & & \vdots \\ -\zeta_k^\dagger z_1 & \cdots & -\zeta_k^\dagger z_k & -\zeta_k^\dagger \zeta_1 & \cdots & -\zeta_k^\dagger \zeta_k \end{bmatrix}$$

K is $N \times N$ ordinary, but B is $2k \times 2k$ super

Equality of Invariants

for all integers $m = 1, 2, 3, \ldots$ we have the identity

$$\mathrm{tr}\, K^m = \mathrm{tr}\,(AA^\dagger)^m = \mathrm{trg}\,(A^\dagger A)^m = \mathrm{trg}\, B^m$$

non–trivial connection between ordinary and superspace

remarkable implication for characteristic function

$$\Phi(\mathrm{tr}\, K, \mathrm{tr}\, K^2, \mathrm{tr}\, K^3, \ldots) = \Phi(\mathrm{trg}\, B, \mathrm{trg}\, B^2, \mathrm{trg}\, B^3, \ldots)$$

same form as function of invariants !!

whole approach will be based on characteristic function

Spectral Decomposition

K and B have the same "relevant" eigenvalues !!

$$K = VYV^\dagger \ , \qquad B = wyw^\dagger$$

with $V \in \mathrm{SU}(N)$ and $w \in \mathrm{U}(k/k)$ as well as

$$Y_n = \begin{cases} y_{p1} & \text{for} \quad n = p, \ p = 1, \ldots, k \\ y_{p2} & \text{for} \quad n = p + k, \ p = 1, \ldots, k \\ * & \text{for} \quad n = 2k + 1, \ldots, N \end{cases}$$

two types of eigenvalues "bosonic" y_{p1} and "fermionic" y_{p2}

Chain of Equalities

characteristic function satisfies

$$\Phi(K) = \Phi(Y) = \Phi(y) = \Phi(B)$$

alternative proof, avoiding the direct use of invariants

Construction of Generating Function

generating function after average over ensemble

$$Z_k(x+J) = \prod_{p=1}^{k} \int d[z_p] \exp\left(i z_p^\dagger (i\varepsilon - x_p + J_p) z_p\right)$$

$$\int d[\zeta_p] \exp\left(i \zeta_p^\dagger (i\varepsilon - x_p - J_p) \zeta_p\right) \Phi(K)$$

insert a proper δ function to rewrite characteristic function

$$\Phi(K) = \Phi(B) = \int d[\rho]\, \Phi(\rho)\, \delta^{(4k^2)}(\rho - B)$$

$$= \int d[\rho]\, \Phi(\rho) \int d[\sigma] \exp\left(-i \operatorname{trg} \sigma(\rho - B)\right)$$

Fourier Superspace Representation

integrals over fields z_p and ζ_p as usual

$$Z_k(x+J) = \int d[\rho]\, \Phi(\rho) \int d[\sigma]\, \exp\left(-i\,\mathrm{trg}\,\sigma\rho\right)\, \mathrm{detg}^{-N}\left(\sigma - x^{-} - J\right)$$

arrive at a Fourier superspace representation only involving the characteristic function

$$Z_k(x+J) = \int d[\rho]\, \exp\left(-i\,\mathrm{trg}\,(x+J)\rho\right)\Phi(\rho) I(\rho)$$

Generalized Ingham–Siegel–type of integral

Fourier transform of superdeterminant to power $-N$

$$I(\rho) = \int d[\sigma] \, \exp\left(i \mathrm{trg}\, \rho\sigma\right) \mathrm{detg}^{\,-N} \sigma^{-}$$

$$= \prod_{p=1}^{k} \Theta(r_{p1})(ir_{p1})^{N} \exp\left(-\varepsilon r_{p1}\right) \frac{\partial^{N-1}\delta(r_{p2})}{\partial r_{p2}^{N-1}}$$

almost equal to superdeterminant to power $+N$

Probability Density in Superspace

convolution theorem in superspace yields

$$Z_k(x+J) \;=\; \int d[\sigma] Q(\sigma) \mathrm{detg}^{-N}(\sigma - x - J)$$

desired probability density is thus Fourier backtransform

$$Q(\sigma) = \int d[\rho]\, \Phi(\rho) \exp\left(-i \mathrm{trg}\, \sigma\rho\right)$$

duality between ordinary and superspace connects Fourier transforms !!

Reduction to Eigenvalue Integrals

Fourier superspace representation has considerable advantages

$\Phi(r)$ and $I(r)$ invariant, apply supersymmetric Harish-Chandra–Itzykson–Zuber integral and do the group integral

$$R_k(x_1, \dots, x_k) = \int d[r]\, B_k(r) \exp\left(-i \operatorname{trg} xr\right) \Phi(r) I(r)$$

with Berezinian (Jacobian) $B_k(r) = \det \left[\frac{1}{r_{p1} - i r_{q2}} \right]_{p,q=1,\dots,k}$

The full problem is reduced to $2k$ integrals, of which k can be done trivially. This holds for arbitrary rotation–invariant probability densities $P(H)$, including those which do not factorize!

General Result beyond Orthogonal Polynomials

another representation for correlation functions

$$R_k(x_1, \ldots, x_k) = \int d[H]\, P(H) R_k^{(\mathrm{fund})}(x - h)$$

$$h = \mathrm{diag}\left(H_{11}, \ldots, H_{kk}, iH_{(k+1)(k+1)}, \ldots, iH_{(2k)(2k)}\right)$$

convolution of probability density with fundamental correlations

$$R_k^{(\mathrm{fund})}(s) = \det\left[C^{(\mathrm{fund})}(s_{p1}, is_{q2})\right]_{p,q=1,\ldots,k}$$

kernel generalizes all Christoffel–Darboux formulae

$$C^{(\mathrm{fund})}(s_{p1}, is_{q2}) = \frac{1}{\pi} \sum_{n=0}^{N-1} \frac{(is_{q2})^n}{(s_{p1}^-)^{n+1}} = \frac{1}{\pi (s_{p1}^-)^N} \frac{(s_{p1}^-)^N - (is_{q2})^N}{s_{p1}^- - is_{q2}}$$

Example

probability density without factorization ($M_1, M_2 = 0, 1, 2, \ldots$)

$$P(H) = \left(\mathrm{tr}\, H^{M_1}\right)^{M_2} \exp\left(-\mathrm{tr}\, H^2\right)$$

correlation functions are linear combinations of determinants

$$R_k(x_1, \ldots, x_k) = \sum_{\{m\}} a_{\{m\}} \sum_{\omega} \det \left[C_{m_{\omega(p)} m_{\omega(k+q)}}(x_p, x_q) \right]_{p,q=1,\ldots,k}$$

$$C_{m_1 m_2}(x_p, x_q) = \exp\left(-x_p^2\right) \sum_{n=0}^{N-1} \frac{1}{n!} \eta_{nm_1}(x_p) \vartheta_{nm_2}(x_q)$$

where $\eta_{nm_1}(x_p)$ and $\vartheta_{nm_2}(x_q)$ are linear combinations of Hermitean polynomials

Summary and Conclusions

- in various applications non–Gaussian probability densities
- Mehta–Mahoux theorem needs factorization
- first step: norm–dependent probability densities
- general case: arbitrary rotation–invariant probability densities
- Fourier superspace formulation only builds upon characteristic function
- all correlation functions reduced to $2k$ (actually k) integrals
- results beyond Mehta–Mahoux theorem
- correlation functions are convolutions involving the fundamental correlations

work in progress with M. Kieburg (Sonderforschungsbereich Transregio 12)