The Riemann-Hilbert Method. A Status Report.

Alexander R. Its
Indiana University-Purdue University
Indianapolis
&
Brunel University
&
Imperial College

Brunel, December, 2007

- **One matrix model** $\equiv$ the asymptotic analysis of orthogonal polynomials on the circle or line and related Toeplitz and Hankel determinants.

- **Two matrix model** $\equiv$ the asymptotic analysis of biorthogonal polynomials.

- **The normal matrix model** $\equiv$ the asymptotic analysis of orthogonal polynomials on the plane

Principal Mathematical References

- G. Szegö
- L. Geronimus
- H. Widom
- E. Basor, C. Tracy, T. Ehrhardt
- E. Saff, V. Totik
- A. Bötcher & B. Silberman
- B. Simon

The Riemann-Hilbert Method

• Solton Theory

Zakharov, Shabat, Manakov, ...

• Quantum Correlation Functions

Jimbo, Miwa, Mori, Sato; Izergin, Korepin, Slavnov, I

• *The Deift-Zhou NSD Method*

• Hankel Determinants, Orthogonal Polynomials on the Line

Fokas, Kitaev, I; Bleher, I; Deift, Kricherbauer, McLaughlin, Venakides, Zhou

• Toeplitz Detereminants, Orthogonal Polynomials on the Circle

Baik, Deift, Johansson

Some of the Recent Important Developments

• Asymptotics of the Partition Function for Hermitian Model via the Riemann-Hilbert Technique

Ercolani, McLaughlin; Bleher, I

• Widom-Dyson Constants

Sine-Kernel: **Krasovsky** (also **Ehrhardt**)

Airy-Kernel: **Deift, Krasovsky, I; Baik, Buckingham, DiFranco**

• Double Scaling Limits and Painlevé Equations

Bleher, I; Bleher, Eynard; Claeys, Duits, Kuijlaars, Vanlessen; Kuijlaars, Östensson, I

(Brézin, Kazakov; Douglas, Shenker; Gross, Migdal; Douglas, Seiberg, Shenker; Crnković, Moore; Periwal, Shevitz; Akemann, Damgaard, Magnea, Nishigaki; Moore; Shcherbina)

• Generalized Fisher-Hartwig Asymptotics

Krasovsky; Krasovsky, I; Deift, Krasovsky, I

• Asymptotics of Generalized Classical Orthogonal Polynomials

M. Vanlessen; Kuijlaars, Martinez-Finkelshtein, Mclaughlin, Saff, Van Assche, Vanlessen

• Multiple Orthogonal Polynomials and Applications. Muli-Matrix Models

Aptekarev, Bleher, Duits, Kuijlaars, Van Assche

• Block-Toeplitz Determinants

Jin, Korepin, I; Mezzadri, Mo, I; (Basor, Ehrhardt)

Toeplitz Determinants

Let $\phi(z)$ be a function defined on the unit circle,

$$C = \{z : |z| = 1\}.$$

The Toeplitz determinant, $D_n[\phi]$, is defined as

$$D_n[\phi] := \det T_n[\phi],$$

where

$$T_n[\phi] := \{\phi_{j-k}\}, \qquad k = 0, ..., n-1,$$

and

$$\phi_k = \int_C \phi(z) z^{-k-1} \frac{dz}{2\pi i}.$$

The question is:

$$D_n[\phi] \sim ?, \qquad n \to +\infty.$$

More generally,

$$\phi(z) \equiv \phi(z|\mathbf{t}),$$

$$D_n \equiv D_n(\mathbf{t}) \sim ?,$$

$$n, \mathbf{t} \to \infty.$$

- Multiple integrals and OPUC.

$$D_n[\phi] = \frac{1}{(2\pi i)^n n!} \int_C \cdots \int_C \prod_{1\le j<k\le n} |z_j - z_k|^2$$

$$\prod_j \frac{\phi(z_j)}{z_j} dz_1 \ldots dz_n$$

$$\frac{D_{n+1}}{D_n} = \frac{1}{2\pi} h_n,$$

$$\int_C p_j(z)\overline{p_k(z)}\phi(z)\frac{dz}{iz} = h_k \delta_{kl},$$

$$p_k(z) = z^k + \ldots.$$

Classical Facts

- Strong Szegö theorem (1958)

$$D_n[\phi] \sim E[\phi](L[\phi])^n, \quad n \to \infty,$$

$$L[\phi] = \exp(\ln \phi)_0,$$

$$E[\phi] = \exp \sum_{k=1}^{\infty} k(\ln \phi)_k (\ln \phi)_{-k}$$

$$\phi(z) - \text{"}good\text{"}.$$

- Fisher-Hartwig asymptotics (1968)

$$\phi(z) = \phi_S(z) \prod_{r=1}^{N} |z - z_r|^{2\alpha_r}, \qquad \alpha_r > -\frac{1}{2},$$

$$D_n \sim E[\phi_S](L[\phi_S])^n n^{\sum_r \alpha_r^2}$$

$$\times \prod_{r=1}^{N} \left(\frac{L(\phi_S)}{\phi_S(z_r)} \right)^{\alpha_r} \frac{G^2(\alpha_r + 1)}{G(2\alpha_r + 1)} \prod_{r \neq s} |z_r - z_s|^{-\alpha_r \alpha_s}.$$

$G(x)$ - Barnes' G-function.

Fisher, **Hartwig**, **Lenard**, **Wu**

Proved by **Widom** (1973).

- Generalization for the case of $\phi(z)$ with jumps

Basor, **Böttcher**, **Silbermann** (1978, 1985)

- Widom's theorem (1971)

$$\phi(z) = \phi_S(z)\chi_{C_\alpha}(z), \quad \phi_S(z) = \phi_S(\bar{z}),$$

$$C_\alpha : \alpha \leq \arg z \leq 2\pi - \alpha.$$

$$D_n \sim E[\psi_S](L[\psi_S])^n n^{-1/4} \left(\cos\frac{\alpha}{2}\right)^{n^2}$$

$$\times \left(\sin\frac{\alpha}{2}\right)^{-1/4} e^{c_0},$$

$$c_0 = 3\zeta'(-1) + \frac{1}{12}\ln 2.$$

Here,

$$\psi_S(e^{i\theta}) = \phi_S\left(e^{2i\,\mathrm{arccos}(\gamma\cos\frac{\theta}{2})}\right).$$

Important particular case, $\phi_S(z) \equiv 1$:

$$D_n \sim \left(\cos\frac{\alpha}{2}\right)^{n^2}\left(n\sin\frac{\alpha}{2}\right)^{-1/4} e^{c_0},$$

$$c_0 = 3\zeta'(-1) + \frac{1}{12}\ln 2.$$

• Widom's theorem for block Toeplitz matrices (1974)

$$\phi(z) - m \times m, \quad m \geq 1$$

$$D_n[\phi] \sim E[\phi](L[\phi])^n, \quad n \to \infty,$$

$$L[\phi] = \mathrm{exp}(\ln \det \phi)_0,$$

$$E[\phi] = \det\left(T_\infty[\phi]T_\infty[\phi^{-1}]\right),$$

$$\phi(z) - \text{``}good\text{''}.$$

Recent developments (based on the RH approach)

• Random permutations (**Baik, Deift, Johansson**) :

$$\phi(z) = e^{t\left(z+\frac{1}{z}\right)}, \quad n, t \to \infty.$$

$$\Longrightarrow$$

$$\lim_{N\to\infty} \text{Prob}\left(\frac{l_N(\sigma) - 2\sqrt{N}}{N^{1/6}} \le s\right) = F_2(s)$$

$l_N(\sigma)$ denotes the length of the longest invreasing subsequence in $\sigma \in S_N$, and $F_2(s)$ is the Tracy-Widom distribution:

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)u^2_{HM}(x)dx\right)$$

• Further generalizations of the Fisher-Hartwig asymptotics (**Krasovsky**)

$$\phi(z) = \phi_S(z) \prod_{r=1}^{N} |z - z_r|^{\alpha_r} \chi_{C_\alpha}(z), \qquad \alpha_r > -\frac{1}{2},$$

$$z_1 = \overline{z_N} = e^{i\alpha}.$$

$$D_n \sim$$

$$E[\psi_S](L[\psi_S])^n n^{2\alpha_1 + \sum_r \alpha_r^2 - 1/4} \left(\cos\frac{\alpha}{2}\right)^{(n+\sum_r \alpha_r)^2}$$

$$\times C_K$$

$$C_K = \left(\sin\frac{\alpha}{2}\right)^{-1/4} \left(\cos\frac{\alpha}{2}\right)^{-(2\alpha_1+\sum_r \alpha_r^2)}$$

$$\times 2^{4\alpha_1+2\alpha_1^2}\Gamma(1+2\alpha_1)\frac{G^2(3/2+2\alpha_1)}{\sqrt{\pi}G(2+4\alpha_1)}$$

$$\times(\sin\alpha)^{2\alpha_1^2}\prod_{r=1}^{N}\left(\frac{L(\psi_S)}{\psi_S(z_r)}\right)^{\alpha_r}$$

$$\times\prod_{r=2}^{N-1}\frac{G^2(\alpha_r+1)}{G(2\alpha_r+1)}\left(\frac{\sin^2(\theta_r/2)}{1-\frac{\cos^2(\theta_r/2)}{\cos^2(\alpha/2)}}\right)^{\alpha_r^2/2}$$

$$\times\prod_{r\neq s}|z_r-z_s|^{-\alpha_r\alpha_s},$$

$$\theta_r \equiv \arg z_r.$$

- The Fredholm determinant representation

$$D_n[\phi] = \det(I - K_n),$$

$$K_n : L_2\,(C; \mathbf{C}) \to L_2\,(C; \mathbf{C})\,,$$

$$K_n(z, z') = \frac{z^n (z')^{-n} - 1}{z - z'} \, \frac{1 - \phi(z')}{2\pi i},$$

(Deift)

Principal observation:

$$K_n(z, z') = \frac{f^T(z)h(z')}{z - z'},$$

$$f(z) = \begin{pmatrix} z^n \\ 1 \end{pmatrix},$$

$$h(z) = \begin{pmatrix} z^{-n} \\ -1 \end{pmatrix} \frac{1 - \phi(z)}{2\pi i},$$

$\Longrightarrow$ K_n is an "integrable" operator $\Longrightarrow$

- The Riemann-Hilbert representation

$$R := (I - K_n)^{-1} K_n,$$

$$R(z, z') = \frac{F^T(z) H(z')}{z - z'},$$

where

$$F(z) = Y_+(z) f(z), \quad H(z) = (Y_+^T)^{-1}(z) h(z),$$

$$z \in C,$$

and the 2×2 matrix function $Y(z)$ is the (unique) solution of the following Riemann-Hilbert problem:

1. $Y(z) \in H(\mathbf{C} \setminus C)$

2. $Y(\infty) = I_2$

3. $Y_-(z) = Y_+(z)G(z), \quad z \in C$

$$G(z) = I_2 + 2\pi i f(z) h^T(z)$$

$$= \begin{pmatrix} 2 - \phi(z) & -z^n(1 - \phi(z)) \\ z^{-n}(1 - \phi(z)) & \phi(z) \end{pmatrix}$$

Remark 1:

$$\frac{D_{n+1}}{D_n} = Y_{11}(0)$$

Remark 2: Let

$$\phi_t(z) := (1-t) + t\phi(z),$$

then

$$\ln D_n[\phi] = -\int_0^1 \left[\int_C \text{trace} \left(\frac{dF_t^T(z)}{dz} H_t(z) \right) dz \right] dt$$

$$F(z) = Y_+(z) \begin{pmatrix} z^n \\ 1 \end{pmatrix}$$

$$H(z) = \left(Y_+^T(z) \right)^{-1} \begin{pmatrix} z^{-n} \\ -1 \end{pmatrix} \frac{1-\phi(z)}{2\pi i}$$

• An alternative RH. Relation to the orthogonal polynomials on the circle.

Observe:

$$G(z) = \begin{pmatrix} 2-\phi(z) & -z^n(1-\phi(z)) \\ z^{-n}(1-\phi(z)) & \phi(z) \end{pmatrix}$$

$$= \begin{pmatrix} z^n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^{-n} & 0 \\ -1 & z^n \end{pmatrix}$$

Put:

$$\tilde{Y}(z) = Y(z)\begin{pmatrix} z^n & -1 \\ 1 & 0 \end{pmatrix}, \quad |z| < 1,$$

$$\tilde{Y}(z) = Y(z)\begin{pmatrix} z^n & 0 \\ 1 & z^{-n} \end{pmatrix}$$

$$\equiv Y(z)\begin{pmatrix} 1 & 0 \\ z^{-n} & 1 \end{pmatrix} z^{n\sigma_3}, \quad |z| < 1,$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

THE NEW RH PROBLEM.

1. $\tilde{Y}(z) \in H(\mathbf{C} \setminus C)$

2. $\tilde{Y}_-(z) = \tilde{Y}_+(z)\tilde{G}(z), \qquad z \in C$

$$\tilde{G}(z) = \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}$$

3. $\tilde{Y}(z)z^{-n\sigma_3} \to I_2, \qquad z \to \infty$

REMARK:

$$\tilde{Y}(z) \to \sigma_3\tilde{Y}(z)\sigma_3$$

$$\tilde{Y}_+(z) = \tilde{Y}_-(z)\begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in C$$

OPUC:

$$\tilde{Y}(z) = \begin{pmatrix} p_n(z) & \frac{1}{2\pi i}\int_C p_n(s)s^{-n}\phi(s)\frac{ds}{s-z} \\ & \\ q_{n-1}(z) & \frac{1}{2\pi i}\int_C q_{n-1}(s)s^{-n}\phi(s)\frac{ds}{s-z} \end{pmatrix}$$

$$\int_C p_k(z)\overline{p_l(z)}\phi(z)\frac{dz}{iz} = h_k\delta_{kl}, \quad p_n(z) = z^n + \ldots,$$

$$q_n(z) = -\frac{2\pi}{\overline{h_n^*}}\,\overline{p_n^*\left(\frac{1}{\bar{z}}\right)}z^n,$$

$p_n^*(z)$ - orthogonal with respect to $\overline{\phi(z)}$.

PRINCIPAL IDEA.

Given $\phi(z)$, find $Y(z)$, such that

1. $Y(z) \in H(\mathbf{C} \setminus C)$

2. $Y_+(z) = Y_-(z)G(z), \quad z \in C$

$$G(z) = \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}$$

3. $Y(z)z^{-n\sigma_3} \to I_2, \quad z \to \infty$

Then

$$Y_{11}(z) = p_n(z),$$

and

$$Y_{12}(0) = \frac{1}{2\pi} h_n \equiv \frac{D_{n+1}}{D_n},$$

where $p_n(z)$ and D_n are, respectively, the OPUC and the Toeplitz determinat associated with the weight $\phi(z)$. (**Baik**, **Deift**, **Johansson**, 1999)

Hankel determinants

Let $\phi(z)$ be a function defined on the real line **R**. The Hankel determinant, $D_n[\phi]$, is defined as

$$D_n[\phi] := \det H_n[\phi],$$

where

$$H_n[\phi] := \{\phi_{j+k}\}, \qquad k = 0, ..., n-1,$$

and

$$\phi_k = \int_{-\infty}^{\infty} z^k \phi(z) dz.$$

The question, again, is:

$$D_n[\phi] \sim ?, \qquad n, \mathrm{t} \to +\infty.$$

- Multiple integrals and OPRL.

$$D_n[\phi] = \frac{1}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j<k\le n}(z_j - z_k)^2$$

$$\prod_{1\le j\le n}\phi(z_j)dz_1\ldots dz_n.$$

$$\frac{D_{n+1}}{D_n} = h_n,$$

$$\int_{-\infty}^{\infty} p_k(z)p_l(z)\phi(z)dz = h_k\delta_{kl},$$

$$p_n(z) = z^n + \ldots,$$

Szegö type asymptotics for finite interval

$$\left\{ \int_{-1}^{1} z^{k+j} \varphi(z)\, dz \right\}_{j,k=0,\dots,n-1}$$

$$\frac{h_n(\varphi)}{h_n(1)} \sim \exp\left\{ \frac{1}{\pi} \int_{-1}^{1} \frac{\ln \varphi(z)}{\sqrt{1-z^2}}\, dz \right\}$$

$n \to \infty$

$$\left(h_n(1) \sim 2^{-n/2} \pi \right)$$

G. Szegö, G. Freud

$$\int_{-1}^{1} \frac{\ln \varphi(z)}{\sqrt{1-z^2}}\, dz > -\infty$$

$$\varphi(z) = (1-z)^{\alpha}(1+z)^{\beta}\omega(z)$$

$\omega(z)$ – real analytic > 0

$\alpha, \beta > -1$

$$\frac{h_n(\omega)}{h_n(1)} \cong \exp\left\{\frac{1}{\pi}\int_{-1}^{1}\frac{\ln\omega(z)}{\sqrt{1-z^2}}\,dz\right\}$$

$$\times\left(1 + \sum_{j=2}^{\infty}\frac{H_j}{n^j}\right), \quad n \to \infty$$

$$\boxed{\frac{D_n(\omega)}{D_n(1)} \sim C\, e^{\frac{n}{\pi}\int_{-1}^{1}\frac{\ln\omega(z)}{\sqrt{1-z^2}}\,dz}}$$

A. Kuijlaars, K. McLaughlin, W. Van Assche,

M. Vanlessen

Szegö type asymptotics for infinite interval

$$\left\{ \int_{-\infty}^{\infty} z^{j+k} \varphi(z)\, dz \right\} \qquad \varphi(z) = e^{-nz^2}$$

$$j,k = 0, \dots, n-1$$

$$D_n^{\text{Hermite}} = n^{-1/12} e^{-n^2\left(\frac{3}{4} + \frac{1}{2}\ln 2\right) + n \ln 2\pi + c_0} \times \left(1 + o(1)\right), \quad n \to \infty$$

$$\left\{ \int_{0}^{\infty} z^{j+k} \varphi(z)\, dz \right\}, \quad \varphi(z) = z^{\nu} e^{-nz}$$

$$j,k = 0, \dots, n-1$$

$$D_n^{\text{Laguerre}} = n^{-1/6 + \nu^2/2} e^{-\frac{3}{2}n^2 + n(\ln 2\pi - \nu) + c_1} \times \left(1 + o(1)\right), \quad n \to \infty$$

$$c_0 = \zeta'(-1)$$

$$c_1 = \frac{4}{3} \ln G(1/2) + \left(1/3 + \nu/2\right) \ln \pi + \left(\nu/2 - 1/18\right) \ln 2 - \ln(1+\nu)$$

Small Perturbation

$$\left\{ \int_0^\infty z^{j+k} \varphi(z)\, dz \right\}_{j,k=0,\ldots,n-1}$$

$$\varphi(z) = z^{\nu} e^{-z} U(z) \quad , \quad \nu \geq -1/2$$

$U(z) - 1$ - Schwartz

$$\frac{D_n(U)}{D_n(1)} = \exp\left\{ n^{1/2} \frac{2}{\pi} \int_0^\infty \ln U(x^2)\, dx + C \right\}$$

$$\times \left(1 + o(1) \right) \quad , \quad n \to \infty$$

$$C = -\frac{\nu}{2} \ln U(0) + \frac{1}{2\pi^2} \int_0^\infty x\, S^2(x)\, dx \, ,$$

$$S(x) = \int_0^\infty \cos(xy) \ln U(y^2)\, dy$$

E. Basor, Y. Chen, H. Widom

Szegö type asymptotics for infinite interval. II

$$\left\{ \int_{-\infty}^{\infty} z^{k+j} \varphi(z) dz \right\}_{k,j=0,\dots,n-1}$$

$$\varphi(z) = e^{-nV(z)} , \quad V(z) = \sum_{j=1}^{2m} t_j z^j \quad , t_{2m} > 0$$

- - - - - - - - - - - - - - - -

Note:

$$e^{-V(z)} \longmapsto e^{-nW(z)}$$

$$z \to n^{1/2m} z$$

$$W(z) = t_{2m} z^{2m} + \sum_{j=1}^{2m-1} \frac{t_j}{n^{1-j/2m}} z^j$$

"Weak Szegö Theorem"
Equilibrium measure

$$\lim_{n\to\infty} \frac{1}{n^2} \ln D_n = \sup_{\mu} \left\{ \iint \ln|z-z'|\, d\mu(z)\, d\mu(z') - \int V(z')\, d\mu(z') \right\}$$

$\boxed{\text{M. Mehta}}$; E. Saff, V. Totik ; K. Johansson, P. Deift, T. Kriecherbauer, K. McLaughlin

$V(z)$ - real analytic $\Rightarrow$ $d\mu = \rho(z)dz$

— — — ... —

D.K.M.

"Strong Szegö Theorem"

$$V(z) = \frac{1}{2} z^2 + \sum_{k=1}^{2m} t_k z^k$$

one interval case

$$\ln \frac{D_n(t)}{D_n(0)} \simeq n^2 F_0(t) + F_1(t) + \frac{1}{n^2} F_2(t) + \cdots$$

D. Bessis, C. Itzykson, J. B. Zuber

N. Ercolani, K. McLaughlin

B. Eynard

. . . .

- Double scaling limit: $e^{-nV(z)}$

$$V(z,t) = \frac{1}{4t^2} z^4 + \left(1 - \frac{2}{t}\right) z^2$$

$t > 1$ - one interval , $t < 1$ - two intervals

$$\frac{D_n(t)}{D_n(\infty)} = F_{TW}\left((t-1)2^{2/3} n^{2/3}\right) e^{n^2 F_0(t) + F_1(t)}$$

$$\times \left(1 + o(1)\right), \quad n \to \infty, \quad (t-1)n^{2/3} = O(1)$$

$$F_{TW}(x) = \exp\left\{ \int_x^\infty (x-y) u^2(y)\, dy \right\}$$

$u(x) = u_{HM}(x)$: $\quad u'' = xu + 2u^3$

$$u(x) \sim Ai(x), \quad x \to +\infty$$

$$\left(u \sim \sqrt{-x/2}, \quad x \to -\infty \right)$$

P. Bleher, A.I.

T. Claeys, A.B.J. Kuijlaars, M. Vanlessen

Fisher-Hartwig asymptotics.
Root type singularity.

$$\left\{ \int_{-\infty}^{\infty} z^{j+k} \varphi(z)\,dz \right\}_{j,k=0,\dots,n-1}$$

$$\varphi(z) = |z-\lambda|^{2\alpha} e^{-2n z^2} \qquad \operatorname{Re}\alpha > -\tfrac{1}{2}$$

$$-1 < \lambda < 1$$

$$\frac{D_n(\alpha)}{D_n(0)} = C(\alpha)\,(1-\lambda^2)^{\alpha^2/2} \left(\frac{n}{2}\right)^{\alpha^2}$$

$$\times\, e^{\alpha n (2\lambda^2 - 1 - 2\ln 2)} \left(1 + O\left(\frac{\ln n}{n}\right)\right)$$

$$C(\alpha) = 2^{2\alpha^2} \frac{G^2(\alpha+1)}{G(2\alpha+1)}$$

P. Forrester, N. Frankel

I. Krasovsky

Fisher - Hartwig asymptotics

Jump

$$\varphi(z) = e^{-2nz^2} \begin{cases} e^{i\beta\pi} & z < \lambda \\ e^{-i\beta\pi} & z > \lambda \end{cases}$$

$$\lambda \in (-1,1) \qquad \beta \notin \mathbb{Z}$$

$$\frac{D_n(\beta)}{D_n(0)} = C(\beta)\,(1-\lambda^2)^{-3\beta^2/2}\,(8n)^{-\beta^2}$$

$$\times\, e^{2in\beta\left(\arcsin\lambda + \lambda\sqrt{1-\lambda^2}\right)}$$

$$\times \left(1 + O\left(\frac{\ln n}{n}\right)\right)$$

$$n \to \infty$$

$$C(\beta) = G(1+\beta)\,G(1-\beta)$$

I. Krasovsky , A.I.

$$\boxed{z P_k(z) = P_{k+1}(z) + \alpha_k P_k(z) + \beta_k P_{k-1}(z)}$$

$$\beta = i\gamma \quad \gamma \in \mathbb{R}$$

$$\alpha_n = \frac{\gamma}{2n\sqrt{1-\lambda^2}}\left[1 + (-1)^n \sin\left(2\gamma \ln n - 2n\varkappa + \theta\right)\right]$$

$$+ O\left(\frac{1}{n^{3/2}}\right), \quad n \to \infty$$

$$\varkappa = \arcsin\lambda + \lambda\sqrt{1-\lambda^2}$$

$$\boxed{\theta} = \frac{3\pi}{2} + 2\gamma \ln 4 + 2\gamma \ln(1-\lambda^2) - \arcsin\lambda$$

$$- 2 \arg \Gamma(i\gamma)$$

I. Krasovsky, A.I.

Y. Chen, G. Pruessner : α_n for $\lambda = 0$, no θ
(2004)

A. Magnus : jump - Jacobi, α_n with θ, conjecture
(1994)

$$h_n \sim \pi e^{-n(1+2\ln 2) + 2i\beta \arcsin\lambda}$$

$$\beta_n \sim \frac{1}{4}$$

Proof.

$$\frac{d}{d\beta} \ln D_n(\beta) = \{ h_n, h_{n-1}, P_n(\lambda), P_{n-1}(\lambda) \}$$

$$\beta_n = \frac{h_n}{h_{n-1}}$$

$$\alpha_n = (2n)^{\frac{n-1}{2}} P_n^2(\lambda) e^{-\lambda^2} \operatorname{sh} \pi \gamma$$

$P_n(z)$, h_n - from the asymptotic analysis of the relevant RH problem.

• The RH problem.

Put:

$$Y(z) = \begin{pmatrix} p_n(z) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} p_n(s)\phi(s)\frac{ds}{s-z} \\ \\ q_{n-1}(z) & \frac{1}{2\pi i}\int_{-\infty}^{\infty} q_{n-1}(s)\phi(s)\frac{ds}{s-z} \end{pmatrix}$$

$$q_n(z) = -\frac{2\pi i}{h_n} p_n(z).$$

Then,

1. $Y(z) \in H(\mathbf{C} \setminus \mathbf{R})$

2. $Y_+(z) = Y_-(z)G(z), \quad z \in \mathbf{R}$

$$G(z) = \begin{pmatrix} 1 & \phi(z) \\ 0 & 1 \end{pmatrix}$$

3. $Y(z)z^{-n\sigma_3} \to I_2, \quad z \to \infty,$

THE PRINCIPAL IDEA:

$$p_n(z) = Y_{11}(z),$$

and

$$h_n = \frac{i}{2\pi} \lim_{z\to\infty} z^{n+1} Y_{12}(z)$$

(**Fokas, Kitaev, I**, 1990)

Asymptotics. The NSD method

$$Y(z) \to \ldots \to Y_0(z):$$

$$||G_0(.) - I||_{L_2 \cap L_\infty} \leq \epsilon_n,$$

$$\epsilon_n \to 0, \quad \text{as} \quad n \to \infty.$$

Typically:

$$\epsilon_n = O\left(\frac{\ln^{\delta_1}}{n^{\delta_2}} e^{-cn^{\delta_3}}\right),$$

(**Manakov** (1973) $\to$ **Deift&Zhou** (1992))

Asymptotics of Toeplitz Determinants. A simple case

(**Deift; Jin, Korepin, I**) Assume that $\phi(z)$ is analytic in a neighborhood of the circle C.

The key identity:

$$G(z) = M(z)\Lambda(z)N(z),$$

where

$$M(z) = \begin{pmatrix} 1 & z^n\left(1-\phi^{-1}(z)\right) \\ 0 & 1 \end{pmatrix}$$

$$N(z) = \begin{pmatrix} 1 & 0 \\ -z^{-n}\left(1-\phi^{-1}(z)\right) & 1 \end{pmatrix},$$

$$\Lambda(z) = \begin{pmatrix} \phi^{-1}(z) & 0 \\ 0 & \phi(z) \end{pmatrix}.$$

Put

$$X(z) = Y(z) \quad \text{if} \quad |z| > 1 + \epsilon, \quad \text{or} \quad |z| < 1 - \epsilon,$$

$$X(z) = Y(z)M(z) \quad \text{if} \quad 1 - \epsilon < |z| < 1,$$

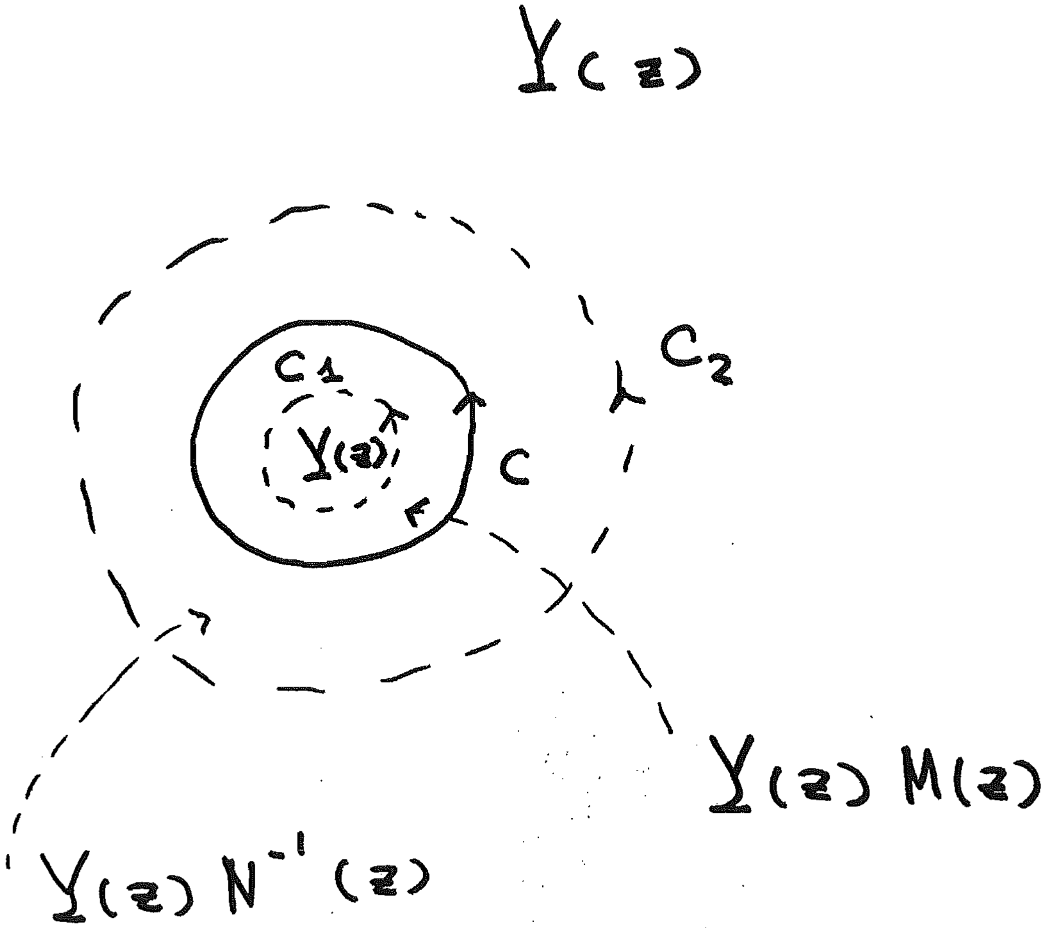
$$X(z) = Y(z)N^{-1}(z) \quad \text{if} \quad 1 < |z| < 1 + \epsilon.$$

Denote

$$C_{1,2} : |z| = 1 \mp \epsilon$$

Put

$X(z)$:



Denote

$$C_{1,2} : |z| = 1 \mp \epsilon$$

$X(z)$ solves the following RH problem,

1^0. $X(z)$ is analytic outside of the contour $\Gamma \equiv C \cup C_1 \cup C_2$

2^0. $X(\infty) = I_2$

3^0. The jumps of the function $X(z)$ across the contour Γ are given by the equations

- $X_-(z) = X_+(z)M(z), \quad z \in C_1$
- $X_-(z) = X_+(z)N(z), \quad z \in C_2$
- $X_-(z) = X_+(z)\Lambda(z), \quad z \in C$

since

$$M(z), N(z) = I_2 + o(1) \quad n \to \infty,$$

we conclude that

$$X(z) \sim X^0(z),$$

where

$$X^0(z) = \begin{pmatrix} u_+(z) & 0 \\ 0 & u_+^{-1}(z) \end{pmatrix}, \quad \text{if} \quad |z| < 1,$$

and

$$X^0(z) = \begin{pmatrix} u_-^{-1}(z) & 0 \\ 0 & u_-(z) \end{pmatrix}, \quad \text{if} \quad |z| > 1.$$

Here:

$$\phi(z) = u_+(z)u_-(z)$$

- the Weiner-Hopf factorization of the symbol $\phi(z)$. From the above analysis, the classical Szegö and Widom's theorems follow.

The Widom-Dyson constant. The Sine kernel case

$$D_n(\alpha) := \det T_n[\phi],$$

where

$$T_n[\phi] := \{\phi_{j-k}\}, \qquad k = 0, ..., n-1,$$

and

$$\phi_k = \int_C \phi(z) z^{-k-1} \frac{dz}{2\pi i},$$

$$\phi(z) = \chi_{C_\alpha}(z),$$

$$C_\alpha : \alpha \leq \arg z \leq 2\pi - \alpha.$$

Widom's Theorem.

$$D_n \sim \left(\cos\frac{\alpha}{2}\right)^{n^2} \left(n \sin\frac{\alpha}{2}\right)^{-1/4}$$

$$\times\, e^{c_0} \qquad , \qquad n \to \infty$$

$$c_0 = 3\,\zeta'(-1) + \frac{1}{12}\ln 2$$

Observation

$$\lim_{n\to\infty} D_n|_{\alpha=2s/n} = \det(1 - K_{\text{sine}}),$$

where

$$K_{\text{sine}} : L_2(0,2s) \to L_2(0,2s),$$

$$K_{\text{sine}}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}.$$

Dyson's conjecture

$$\det(1 - K_{\text{sine}}) \sim e^{-\frac{s^2}{2}} s^{-\frac{1}{4}}$$

$$\times e^{3\zeta'(-1)+\frac{1}{12}\ln 2}, \qquad s \to \infty. \qquad (1)$$

(proven by **Krasovsky** and **Ehrhardt**)

Krasovsky :

$$D_n = \left(\cos\frac{\alpha}{2}\right)^{n^2}\left(n\sin\alpha/2\right)^{-1/4}$$

$$\times e^{c_0}\left(1 + O\left(\frac{1}{n\sin\alpha/2}\right)\right)$$

$$n \to \infty$$

$$\frac{2s}{n} \le \alpha < \pi$$

$$s > s_0$$

The third proof

(**Deift, Krasovsky, Zhou, I**)

The RH problem to be solved is the following one.

- $Y(z)$ is holomorphic for all $z \notin \Gamma_\alpha$

- $Y(\infty) = I$

- $Y_-(z) = Y_+(z) \begin{pmatrix} 2 & -z^n \\ z^{-n} & 0 \end{pmatrix}, \qquad z \in \Gamma_\alpha$

$$\Gamma_\alpha = \{z : -\alpha \leq \arg z \leq \alpha\}.$$

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) = -\frac{n^2}{\sin^2 \alpha} [Y_{12}(0)]^2.$$

$$\begin{pmatrix} 2 & -z^n \\ z^{-n} & 0 \end{pmatrix} \neq \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

g - **function:**

$$g(z) = \frac{z + 1 + \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2z}$$

- $g(z)$ is holomorphic for all $z \notin \Gamma_\alpha$
- $g(\infty) = 1$
- $\left|\frac{g_+(z)}{g_-(z)}\right| < 1, \quad z \neq e^{\pm i\alpha}$
- $g_+(z) g_-(z) = \frac{\kappa}{z}, \quad \kappa = \cos^2 \frac{\alpha}{2}$

Put

$$X(z) = \kappa^{-\frac{n}{2}\sigma_3} Y(z) (g(z))^{-n\sigma_3} \kappa^{\frac{n}{2}\sigma_3}$$

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

- $X(z)$ is holomorphic for all $z \notin \Gamma_\alpha$

- $X(\infty) = I$

- $X_-(z) = X_+(z) \begin{pmatrix} 2\left(\frac{g_+(z)}{g_-(z)}\right)^n & -1 \\ 1 & 0 \end{pmatrix}$

  $z \in \Gamma_\alpha$

Since $\left|\frac{g_+(z)}{g_-(z)}\right| < 1, \quad z \neq e^{\pm i\alpha}$, we expect that

$$X(z) \sim X^0(z):$$

- $X^0(z)$ is holomorphic for all $z \notin \Gamma_\alpha$
- $X^0(\infty) = I$
- $X^0_-(z) = X^0_+(z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

  $z \in \Gamma_\alpha$

SOLUTION to the MODEL PROBLEM:

$$X^0(z) = \begin{pmatrix} \frac{\delta+\delta^{-1}}{2} & \frac{\delta-\delta^{-1}}{2i} \\ -\frac{\delta-\delta^{-1}}{2i} & \frac{\delta+\delta^{-1}}{2} \end{pmatrix},$$

where

$$\delta(z) = \left(\frac{z - e^{i\alpha}}{z + e^{-i\alpha}}\right)^{\frac{1}{4}}.$$

The leading term of Widom's asymptotics, i.e.

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) \sim -\frac{n^2}{4} \sec^2 \frac{\alpha}{2}, \qquad (2)$$

(formally) follows. (**Deift**, **Zhou**, **I**; 1997)

• More carefull analysis (see Appendix) along the lines indicated allows to obtain the following (rigorous) extension of (2).

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) = -\frac{n^2}{\sin^2 \alpha} \Delta(n, \alpha), \qquad (3)$$

$$\Delta(n, \alpha) = \sin^2 \frac{\alpha}{2} - \frac{\cos^2(\alpha/2)}{4n^2}$$
$$+ O\left(\frac{1}{n^3 \sin^3(\alpha/2)}\right) \sin^2 \alpha,$$

$$n \to \infty, \qquad \frac{2s}{n} \le \alpha \le \pi, \qquad s \ge s_0.$$

<u>Multiple integral:</u>

$$D_n(\alpha)$$

$$= \frac{1}{(2\pi)^n n!} \int_{\alpha}^{2\pi-\alpha} \cdots \int_{\alpha}^{2\pi-\alpha} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \, d\theta_1 \ldots d\theta_n$$

$$= \frac{1}{(2\pi)^n n!} (\pi-\alpha)^{n^2} \left\{ \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1 \le j < k \le n} (x_j - x_k)^2 \, dx_1 \ldots dx_n \right.$$

$$\left. + O((\pi-\alpha)^2) \right\}$$

$$= \frac{1}{(2\pi)^n n!} (\pi-\alpha)^{n^2} \left\{ n! \prod_{k=0}^{n-1} h_k + O((\pi-\alpha)^2) \right\}$$

$$h_k = \frac{2^{2k} (k!)^4}{[(2k)!]^2} \frac{2}{2k+1}$$ Legendre !

• From the multiple integral representation of $D_n(\alpha)$ one obtains

$$\ln D_n(\alpha) = n^2 \ln(\pi - \alpha) - n \ln 2\pi$$

$$+ \ln A_n + O((\pi - \alpha)^2), \qquad (4)$$

where

$$A_n = \prod_{k=0}^{n-1} \frac{2^{2k}(k!)^4}{[(2k)!]^2} \frac{2}{2k+1}$$

$$= e^{c_0} n^{-1/4} (2\pi)^n 2^{-n^2} (1 + o(1)), \quad n \to \infty,$$

$$c_0 = 3\zeta'(-1) + \frac{1}{12} \ln 2.$$

(3) and (4) imply (1)

(**Deift, Krasovsky, Zhou, I**)

REMARK:

$$\det(1 - K_{\text{sine}}) = \exp\left(\int_0^s \frac{\sigma_V(t)}{t} dt\right)$$

where $\sigma_V(s)$ - the special solution of P_V:

$$(s\sigma'')^2 + 4(4\sigma - s\sigma' - (\sigma')^2)(\sigma - s\sigma') = 0,$$

$$\sigma(s) \sim -\frac{2}{\pi} s, \quad s \to 0.$$

(**JMMS**)

The Dyson - Ehrhardt - Krasovsky formula implies that

$$v.p. \int_0^\infty \frac{\sigma_V(t)}{t} dt = 3\zeta'(-1) + \frac{1}{12} \ln 2$$

Appendix

Put

$$\Phi(\lambda) := X^{-1}(-1)X(z(\lambda)),$$

where

$$z(\lambda) = \frac{1 + i\lambda \tan\frac{\alpha}{2}}{1 - i\lambda \tan\frac{\alpha}{2}}$$

maps the interval $[-1, 1]$ to the arc Γ_α.

- $\Phi(\lambda)$ is holomorphic for all $\lambda \notin [-1, 1]$

- $\Phi(\infty) = I$

- $$\Phi_-(\lambda) = \Phi_+(\lambda) \begin{pmatrix} 2\left[\frac{1-\sqrt{1-\lambda^2}\sin\frac{\alpha}{2}}{1+\sqrt{1-\lambda^2}\sin\frac{\alpha}{2}}\right]^n & -1 \\ 1 & 0 \end{pmatrix},$$

  $$\lambda \in (-1, 1)$$

Note:

- The Φ - problem is regular in the neighborhood of $\alpha = \pi$

- The following relation takes place

$$\frac{d^2}{d\alpha^2} \ln D_n(\alpha) = -\frac{n^2}{\sin^2 \alpha} \Delta(n, \alpha),$$

where

$$\Delta(n, \alpha) = \left[\Phi^{-1}(-i \cot \frac{\alpha}{2}; n, \alpha) \Phi(i \cot \frac{\alpha}{2}; n, \alpha) \right]_{12}^2 .$$

- D_n satisfies Painlevé VI equation:

$$\eta(t) \equiv t(t-1)\frac{d}{dt}\ln D_n, \qquad t \equiv e^{-2i\alpha}$$

$$\left(\frac{d\eta}{dt}\right)^4 = \left(\frac{d\eta}{dt} - \frac{n^2}{4}\right)\left(t(t-1)\frac{d^2\eta}{dt^2}\right)^2$$

$$+\left[2\left(\frac{d\eta}{dt} - \frac{n^2}{4}\right)\left(t\frac{d\eta}{dt} - \eta\right) - \left(\frac{d\eta}{dt}\right)^2 + \frac{n^2}{2}\frac{d\eta}{dt}\right]^2.$$

$$\Delta = \frac{1-t}{n^2}\frac{d\eta}{dt} + \frac{1}{n^2}\eta.$$

(Deift, Zhou, I; Tracy, Widom)

Asymptotics of the Hankel with the jump

- $$Y_-(z) = Y_+(z) \begin{pmatrix} 1 & e^{-2nz^2}\omega(z) \\ 0 & 1 \end{pmatrix}$$

- $$Y(z) = (I + O(\tfrac{1}{z}))\, z^{n\sigma_3} \qquad z \to \infty$$

$$\omega(z) = \begin{cases} e^{i\beta\pi} & z < \lambda \\ e^{-i\beta\pi} & z > \lambda \end{cases}$$

$$Y(z) \xrightarrow{1} T(z) \xrightarrow{2} S(z) \xrightarrow{3} S_0(z)$$

$$\longmapsto Y_0(z) = S(z)\, S_0^{-1}(z)$$

1. **g - function.**

$$g(z) = -2\int_1^z \sqrt{s^2-1}\, ds + z^2 + \frac{\ell}{2}, \quad \ell = -1-2\ln 2$$

- $g(z) = \ln z + O(1/z), \quad z \to \infty$

- $$g_+(z) + g_-(z) - 2z^2 - \ell = \begin{cases} 0 & z \in (-1,1) \\ <0 & z \in \mathbb{R}\setminus[-1,1] \end{cases}$$

- $$g_+(z) - g_-(z) = \begin{cases} 0 & z > 1 \\ 2\pi i & z < -1 \end{cases}$$

$$h(z) \equiv g_+(z) - g_-(z) = -4i\int_1^z \sqrt{1-s^2}\, ds, \quad z \in (-1,1)$$

$\mathrm{Re}\, h > 0 \quad \uparrow$

$\mathrm{Re}\, h < 0 \quad \downarrow$

$$G_T(z)$$

$$= e^{n(g_-(z)-\ell/2)\sigma_3} \begin{pmatrix} 1 & \omega e^{-2nz^2} \\ 0 & 1 \end{pmatrix} e^{-n(g_+(z)-\ell/2)\sigma_3}$$

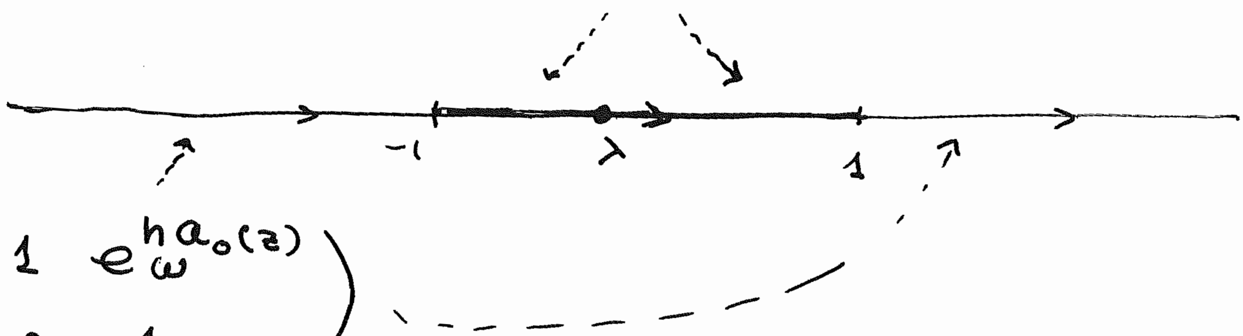
$$= \begin{pmatrix} e^{n(g_- - g_+)} & \omega e^{n(g_+ + g_- - 2z^2 - \ell)} \\ 0 & e^{-n(g_- - g_+)} \end{pmatrix}$$

$$T(z) := e^{-n\ell\sigma_3/2}\, Y(z)\, e^{-n\left(g(z)-\ell/2\right)\sigma_3} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $T(z) = I + O(1/z)\,, \quad z \to \infty$

- $T_+ = T_-\, G_T(z)$:

$$\begin{pmatrix} e^{-nh} & \omega \\ 0 & e^{nh} \end{pmatrix}$$



$$\begin{pmatrix} 1 & e^{n a_0(z)}_{\omega} \\ 0 & 1 \end{pmatrix}$$

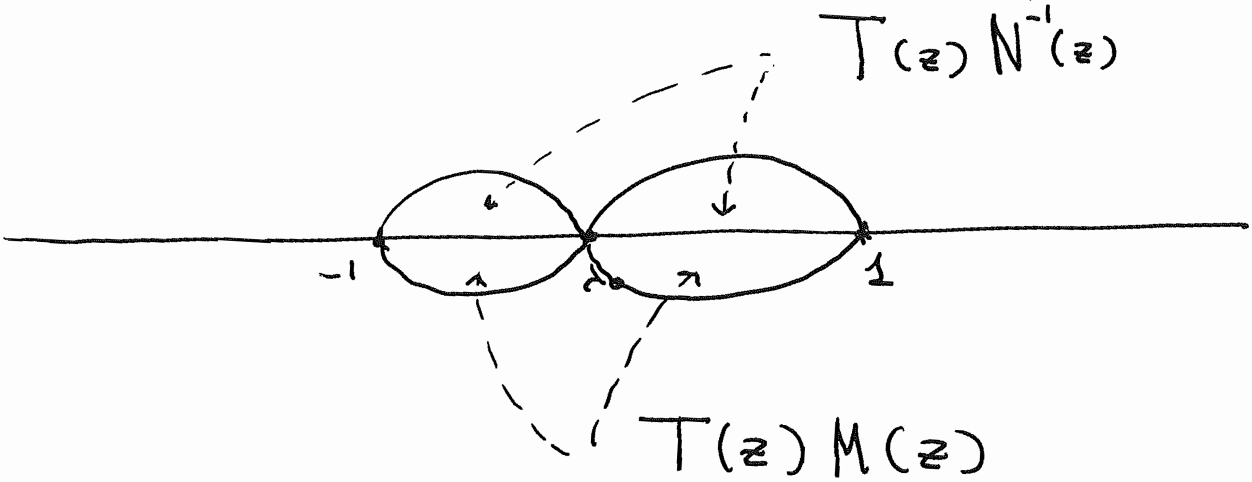
$$\omega = \begin{cases} e^{-i\beta\pi} & z > \lambda \\ e^{+i\beta\pi} & z < \lambda \end{cases}$$

$$a_0(z) = g_+ + g_- - 2z^2 - \ell = -4\int_1^z \sqrt{s^2-1}\, ds$$

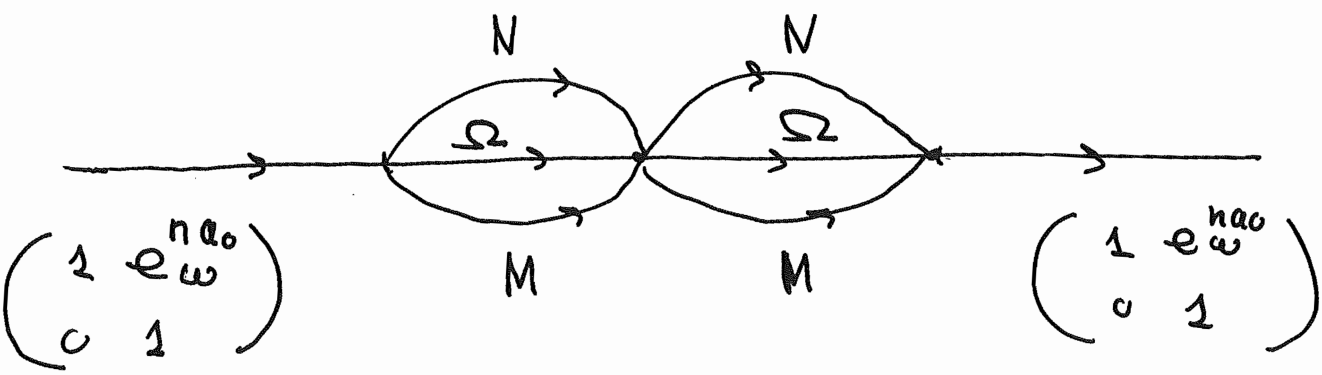
2. Opening of the lenses.

$$\begin{pmatrix} e^{-nh} & \omega \\ 0 & e^{nh} \end{pmatrix} = \underset{M}{\begin{pmatrix} 1 & 0 \\ \omega^{-1}e^{nh} & 1 \end{pmatrix}} \underset{\Omega}{\begin{pmatrix} 0 & \omega \\ -\omega^{-1} & 0 \end{pmatrix}} \underset{N}{\begin{pmatrix} 1 & 0 \\ \omega^{-1}e^{-nh} & 1 \end{pmatrix}}$$

$\underline{S(z)}$:



$\underline{G_S(z)}$:

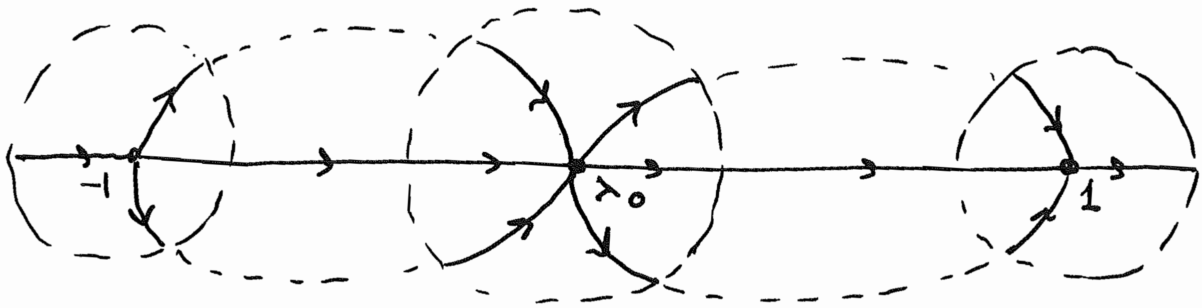


3. Model RH problems (parametrix)

intuitively: $\quad S(z) \sim S^{(\infty)}(z)$:

- $S^{(\infty)}_+(z) = S^{(\infty)}_-(z)\,\Omega(z)$

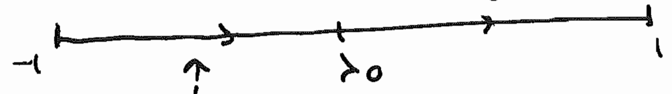
- $S^{(\infty)}(\infty) = I$.



$$S_0(z) := \begin{cases} S^{(\infty)}(z) & z \in \mathbb{C} \setminus U_{-1} \cup U_{\lambda_0} \cup U_1 \\ S^{(\pm 1)}(z) & z \in U_{\pm 1} \\ S^{(\lambda_0)}(z) & z \in U_{\lambda_0} \end{cases}$$

$\boxed{S^{(\infty)}(z):}$

$$\begin{pmatrix} 0 & e^{-i\beta\pi} \\ -e^{+i\beta\pi} & 0 \end{pmatrix}$$



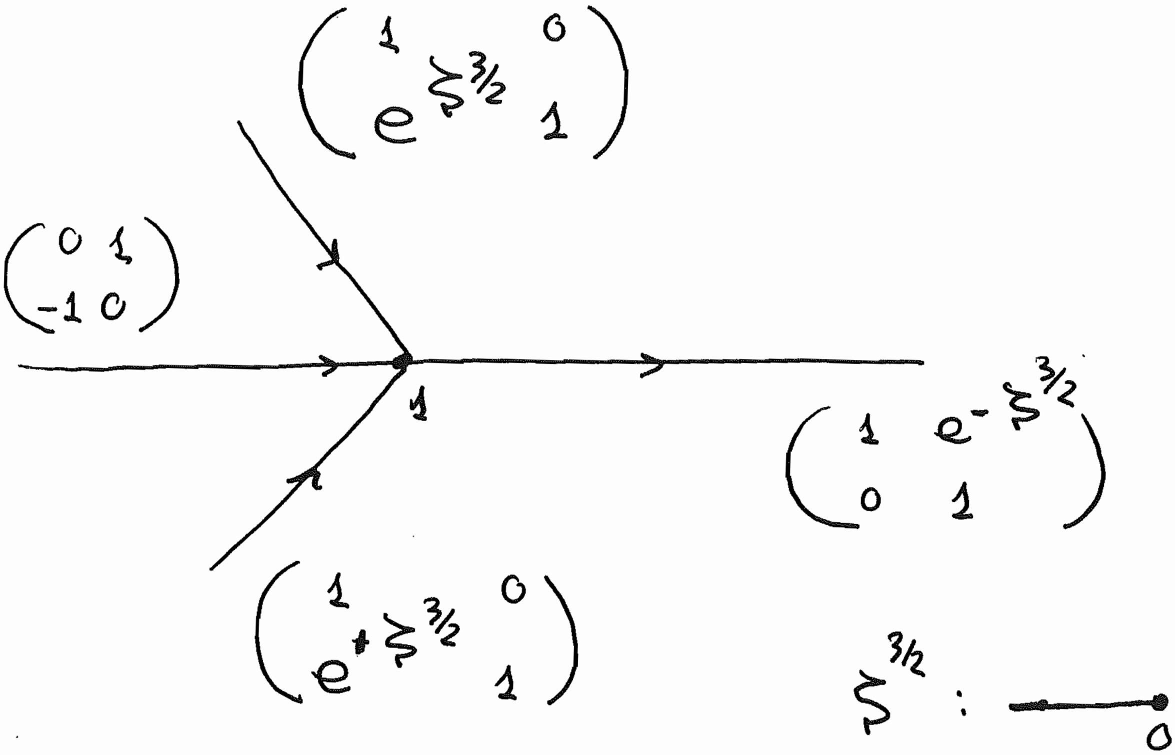
$$\begin{pmatrix} 0 & e^{+i\beta\pi} \\ -e^{-i\beta\pi} & 0 \end{pmatrix}$$

$$\frac{1}{2} D_\infty^{\sigma_3} \begin{pmatrix} a + a^{-1} & -i\,(a - a^{-1}) \\ i\,(a - a^{-1}) & a + a^{-1} \end{pmatrix} \mathcal{D}^{-\sigma_3}(z)$$

$$a(z) = \left(\frac{z-1}{z+1}\right)^{1/4} \qquad D_\infty = \lim_{z\to\infty} \mathcal{D}(z)$$

$$\mathcal{D}(z) = \exp\left\{ \frac{\sqrt{z^2-1}}{2\pi} \int_{-1}^{1} \frac{\ln \omega}{\sqrt{1-s^2}} \frac{ds}{z-s} \right\}$$

$\boxed{S^{(\pm 1)}(z):}$

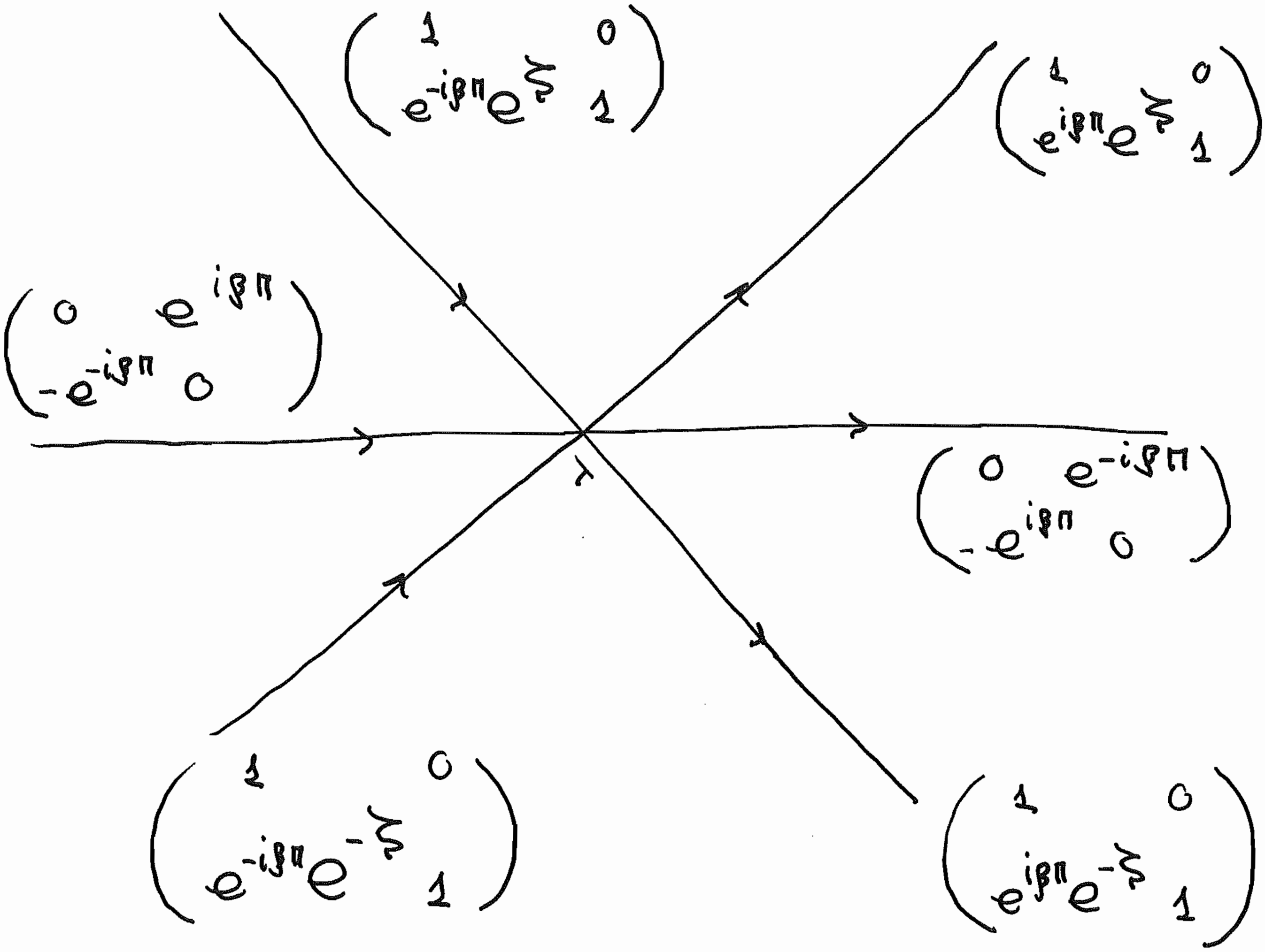


$$\zeta = \left(4n \int_1^z \sqrt{s^2 - 1}\, ds \right)^{3/2} = c_0 (z-1) + \cdots\cdots$$

This is the $\boxed{\text{Airy - RH problem:}}$

solvable in terms of $Ai(\zeta)$

$\boxed{S^{(\lambda_0)}(z):}$



$$\zeta = 4in \int_{\lambda}^{z} \sqrt{1-s^2}\, ds = 4in\sqrt{1-\lambda^2}\,(z-\lambda_0)+\cdots$$

This is th CH- RH problem:

$$G(\zeta) = e^{-\zeta/2\,\sigma_3}\, G(0)\, e^{\zeta/2\,\sigma_3}$$

Solvable in terms of the confluent hypergeometric function $\Psi(\beta, 1, \zeta)$