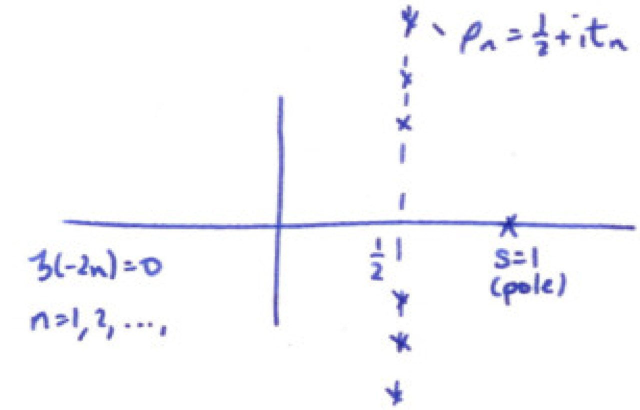
Brunel Talk

$$\zeta(s) = \prod_p \left(1-\frac{1}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \mathrm{Re}\, s > 1$$



RH : $t_n \in \mathbb{R} \ \forall n$

mean density $\sim \frac{1}{2\pi} \log \frac{|t|}{2\pi}$

Montgomery's conjecture (1973)

$$\lim_{t\to\infty} \left(\begin{array}{c}\text{pair correlation of} \\ t_n \simeq t\end{array}\right) = \lim_{N\to\infty} \left(CUE_N / GUE_N \text{ pair correlation}\right)$$

$$= \delta(x) + 1 - \frac{\sin^2 \pi x}{\pi^2 x^2}$$

Moment conjecture (K-Snaith 2000)

$$\frac{1}{T}\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2\lambda} dt \underset{T\to\infty}{\sim} a(\lambda) \left\langle \left|\det(I - A e^{-i\theta})\right|^{2\lambda} \right\rangle_{\substack{A \in CUE_N \\ N \sim \log \frac{T}{2\pi}}}$$

$$a(\lambda) = \prod_p \left[\left(1-\frac{1}{p}\right)^{\lambda^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+\lambda)}{m!\,\Gamma(\lambda)}\right)^2 p^{-m}\right]$$

$N \to \infty$

$$\cancel{\log}\ N^{\lambda^2} \frac{G^2(1+\lambda)}{G(1+2\lambda)}$$

$$= N^{k^2} \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \quad \text{if } \lambda = k \in \mathbb{N}$$

G = Barnes' G-function

Lower Order Terms in the moment conjecture

(CFKRS)

$$\langle |\det(I-Ae^{-i\theta})|^{2k} \rangle_{A\in CUE_N} = \prod_{j=0}^{k-1} \frac{j!}{(j+k)!} \prod_{i=1}^{k} (N+i+j)$$

$$= Q_k(N)$$

polynomial of degree k^2

conjecture: $\frac{1}{T}\int_0^T |\zeta(\tfrac12+it)|^{2k}\,dt = P_k(\log \frac{T}{2\pi}) + o(1)$

polynomial of degree k^2

$\boxed{\text{slides}}$

Hybrid Formulae (Gonek, Hughes, K 2007)

$$\zeta(\tfrac12+it) \text{ “=” } \prod_p \left(1-\frac{1}{p^{1/2+it}}\right)^{-1} \rightarrow a(\lambda)$$

$$= \frac{e^{(\log 2\pi - 1 - \frac12\gamma)s}}{2(s-1)\Gamma(\frac{s}{2}+1)} \prod_n \left(1-\frac{s}{\frac12+it_n}\right) e^{\frac{s}{\frac12+it_n}} \Bigg|_{s=\frac12+it}$$

RMT factor

$$= \ldots \prod_{t_n} (\ldots)$$

$$= \ldots \prod_{|t-t_n|\le y} (\ldots) \prod_{|t_y-t_n|>y} (\ldots)$$

$$= \dots \prod_{|t-t_n|\le y} (\dots) \prod_{p} (\quad)_y$$

$\boxed{\text{SLIDE}}$

$$\zeta(\tfrac{1}{2}+it) = Z_X(\tfrac{1}{2}+it)\, P_X(\tfrac{1}{2}+it) + \text{small error}$$

$$Z_X(\tfrac{1}{2}+it) \simeq e^{-\sum_n E_1(i(t-t_n)\log X)} \simeq \prod_{|t-t_n|e^{\gamma}\log X \le 1} (i(t-t_n)e^{\gamma}\log X)$$

$$P_X(\tfrac{1}{2}+it) \simeq \prod_{p\le X} \left(1 - \frac{1}{p^{1/2+it}}\right)^{-1}$$

$\boxed{\text{PICTURES}}$

Th^{m}/ Let $X \to \infty$ and $T \to \infty$ st. $X = O((\log T)^{2-\varepsilon})$.

then

$$\frac{1}{T}\int_T^{2T} |P_X(\tfrac{1}{2}+it)|^{2k}\, dt \sim a(k)\, (e^{\gamma}\log X)^{k^2}$$

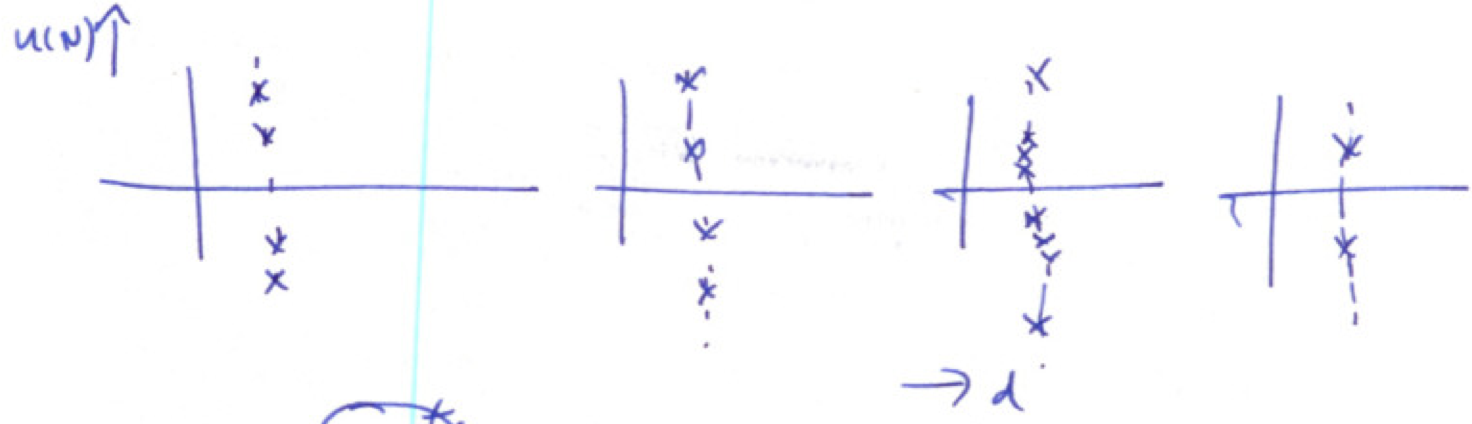
CUE_N average of $\quad |Z_X(\tfrac{1}{2}+it)|^{2k} \sim \dfrac{G^2(1+k)}{G(1+2k)} \left(\dfrac{\log T}{e^{\gamma}\log X}\right)^{\lambda^2}$

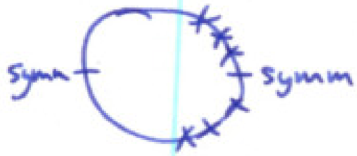
$N \sim \log \frac{T}{2\pi}$

⌈Heine's identity + Fisher-Hartwig asymptotics⌋

$\rightarrow$ original moment conjecture.

Families of L-functions (Katz-Sarnak)





$\longrightarrow U(N),\ USp(2N),\ O(N)$

eg1

$$\chi_d(p) = \begin{cases} +1 & \text{if } p \nmid d \text{ and } x^2 \equiv d \pmod p \text{ soluble} \\ -1 & \quad " \qquad\qquad\qquad " \quad \text{not soluble} \\ 0 & \text{if } p \mid d \end{cases}$$

$$L_D(s, \chi_d) = \prod_p \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}$$

symplectic family.

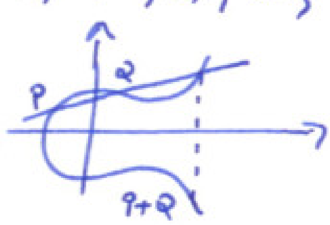
eg2 Elliptic curve L-functions

$E: \quad y^2 = x^3 + ax + b \qquad a, b \in \mathbb{Z}$

$E(\mathbb{Q}) = \{$ rational solutions of $E\} \cup \{\infty\}$

eg $y^2 = x^3 - x \qquad E(\mathbb{Q}) = \{(0,0), (1,0), (-1,0), \infty\}$

Mordell: $E(\mathbb{Q})$ Abelian group



Mordell: $E(\mathbb{Q})$ is finitely generated

$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{tors}$$

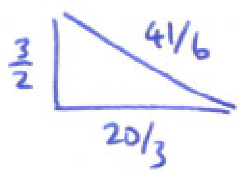
... of elements of finite

$\boxed{\text{what is the distribution of ranks?}}$

eg $E_d: \quad dy^2 = x^3 + ax + b$

what is distribution of rank(E_d)?

example is $d \in \mathbb{N}$ the area of a rational right triangle?



area $= 5$

yes if rank$(dy^2 = x^3 - x) > 0$!

L-functions

elliptic curve $\longleftrightarrow$ modular form

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z)$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \text{ st } k \mid c$$

fourier coefficients a_n

$$\zeta_E(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$L_E(s, \chi_d) = \zeta_{E_d}(s) =$$

$\boxed{\text{orthogonal family}}$

Birch-Swinnerton-Dyer conjecture

$\boxed{\text{order of vanishing of } L_E(1, \chi_d) = \text{rank of } E_d}$

rmt + arithmetic $\Rightarrow$

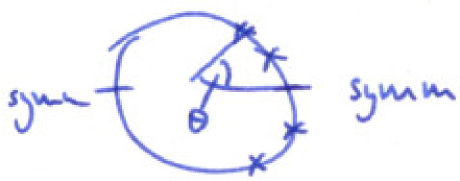
$$\#\{p < D : \text{rank}(E_p) > 0 \text{ nontrivially}\} \simeq \frac{D^{3/4}}{(\log D)^{5/8}}$$

(CKRS)

Function-field L-functions

- polynomials
- RH is true!
- RMT is true
- G_2 (K-L-R)
- other exceptional groups?

Symmetry Transitions? (K-Odgers)



O or Sp at symm points
Unitary far from symm points?

local statistics – transition on scale of mean spacing

moments – depend on θ!

eg $$\left\langle |\det(I - Ae^{-i\theta})|^{2k} \right\rangle_{USp(2N)} \underset{N\to\infty}{\sim} \frac{\left\langle |\det(I - Ae^{-i\theta})|^{2k} \right\rangle_{U(2N)}}{(2\sin\theta)^{k(k+1)}}$$