Entanglement entropy in quantum spin chains with finite range interaction

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Work in collaboration with Alexander Its and Man Yue Mo

Outline

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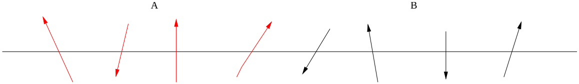

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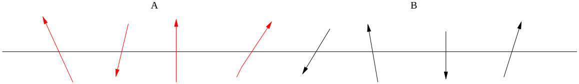

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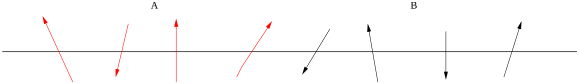

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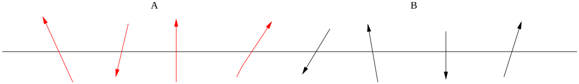

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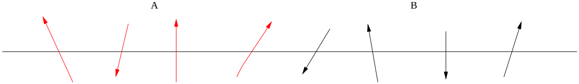

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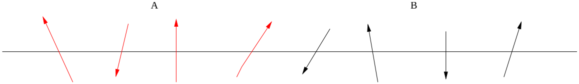

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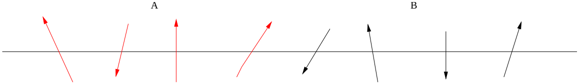

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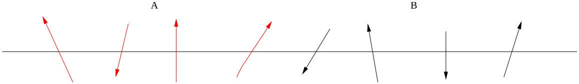

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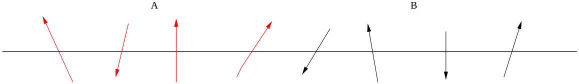

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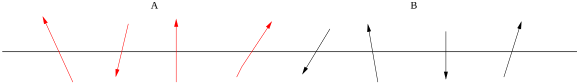

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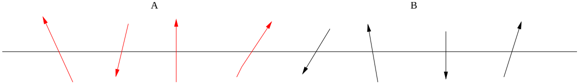

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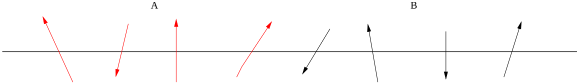

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- If the model is: a) integrable; b) the interaction is translation invariant.
- Many physical quantities can be computed in terms of matrices with a Toeplitz structure. (In general block-Toeplitz matrices.)

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- Consider Hamiltonians that can be transformed into

$$H_\alpha = \alpha \left[\sum_{j,k=0}^{M-1} b_j^\dagger A_{j-k} b_k + \frac{\gamma}{2} \left(b_j^\dagger B_{j-k} b_k^\dagger - b_j B_{j-k} b_k \right) \right] - 2 \sum_{j=0}^{M-1} b_j^\dagger b_j .$$

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- In terms of Pauli determinants it becomes

$$H_\alpha = -\frac{\alpha}{2} \sum_{0 \le j \le k \le M-1} \left[(A_{j-k} + \gamma B_{j-k}) \sigma_j^x \sigma_k^x \left(\prod_{l=j+1}^{k-1} \sigma_l^z \right) + (A_{j-k} - \gamma B_{j-k}) \sigma_j^y \, \sigma_k^y \left(\prod_{l=j+1}^{k-1} \sigma_l^z \right) \right] - \sum_{j=0}^{M-1} \sigma_j^z .$$

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Spin chains and block-Toeplitz determinants

- The entropy of the entanglement can be written as

$$S(\rho_A) = \lim_{\epsilon \to 0^+} \left[\lim_{L \to \infty} \frac{1}{4\pi i} \oint_{\Gamma(\epsilon)} e(1+\epsilon, \lambda) \times \frac{d}{d\lambda} \log \left(D_L(\lambda)(\lambda^2 - 1)^{-L} \right) d\lambda \right],$$

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where

$$e(x,\nu) = -\frac{x+\nu}{2} \log\left(\frac{x+\nu}{2}\right) - \frac{x-\nu}{2} \log\left(\frac{x-\nu}{2}\right).$$

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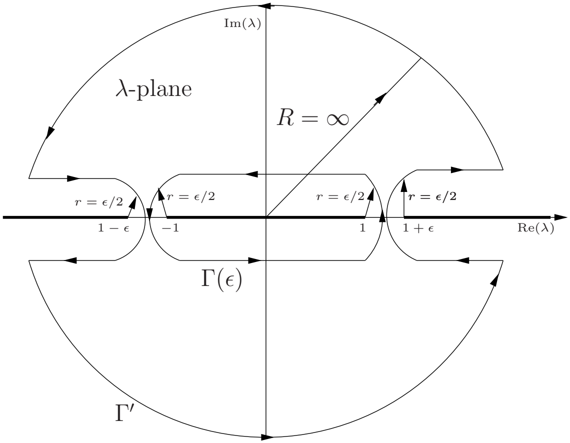

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$D_L(\lambda)$ is the determinant of the block-Toeplitz matrix

$$T_L[\mathbf{\Phi}] = \begin{pmatrix} B_0 & B_{-1} & \cdots & B_{2-L} & B_{1-L} \\ B_1 & B_0 & \cdots & B_{3-L} & B_{2-L} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{L-2} & B_{L-3} & \cdots & B_0 & B_{-1} \\ B_{L-1} & B_{L-2} & \cdots & B_1 & B_0 \end{pmatrix}$$

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- n is the length of the interaction.
- Define

$$\{\lambda_1, \lambda_2, \ldots, \lambda_{4n}\} = \{z_1, \ldots, z_{2n}, z_1^{-1}, \ldots, z_{2n}^{-1}\}.$$

Spin chains and block-Toeplitz determinants

Remarks:

- The branch cuts of $g(z)$ are the segments

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- The genus of $\mathcal{L}$ is $g = 2n - 1$.

Statement of results

- Define $\theta : \mathbb{C}^g \longrightarrow \mathbb{C}$ associated to $\mathcal{L}$ by

$$\theta(\vec{s}) := \sum_{\vec{n} \in \mathbb{Z}^g} e^{i\pi \vec{n} \cdot \Pi \vec{n} - 2\pi i \vec{s} \cdot \vec{n}}.$$

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Theorem. (AR Its, FM and MY Mo, 2007)

As $L \to \infty$ the von Neumann entropy is

$$S(\rho_A) = \frac{1}{2} \int_1^\infty \log \frac{\theta\left(\beta(\lambda)\overrightarrow{e} + \frac{\tau}{2}\right) \theta\left(\beta(\lambda)\overrightarrow{e} - \frac{\tau}{2}\right)}{\theta^2\left(\frac{\tau}{2}\right)} d\lambda,$$

where $\overrightarrow{e}$ is a $2n-1$ vector whose last n entries are 1 and the first $n-1$ entries are 0 and τ depends on $\mathcal{L}$.

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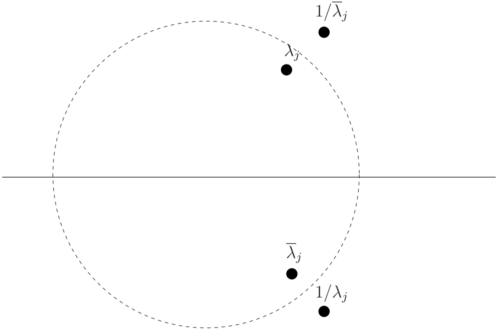
- Since the z_js are roots of a polynomial with real coefficient



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Theorem. (AR Its, FM and MY Mo, 2007)
Let the m pairs of roots $\lambda_j, \overline{\lambda}_j^{-1}$, $j = 1, \dots, m$, approach together towards the unit circle such that the limiting values of $\lambda_j, \overline{\lambda}_j^{-1}$ are distinct from those of $\lambda_k, \overline{\lambda}_k^{-1}$ if $j \neq k$, then the entanglement entropy is asymptotic to

$$S(\rho_A) = -\frac{1}{6}\sum_{j=1}^{m} \log\left|\lambda_j - \overline{\lambda}_j^{-1}\right| + O(1),$$

$$\lambda_j \to \overline{\lambda}_j^{-1}, \quad j = 1, \dots, m.$$

Block-Toeplitz determinants and the RH problem

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- The symbol $\mathbf{\Phi}(z) = \begin{pmatrix} i\lambda & g(z) \\ -g^{-1}(z) & i\lambda \end{pmatrix}$ admits the Wiener-Hopf factorization:

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where $U_\pm(z)$ and $V_\pm(z)$ are analytic inside/outside the unit circle and

$$U_-(\infty) = V_-(\infty) = I.$$

Block-Toeplitz determinants and the RH problem

Theorem (Widom, 1974)

$$\frac{d}{d\lambda}\log D_L(\lambda) = -\frac{2\lambda}{1-\lambda^2}L$$
$$+\frac{1}{2\pi}\int_{\mathbb{S}^1} \operatorname{tr}\left[\left(U_+'(z)U_+^{-1}(z) + V_+^{-1}(z)V_+'(z)\right)\Phi^{-1}(z)\right]dz + r_L(\lambda),$$

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where

$$|r_L(\lambda)| \le \frac{C}{|\lambda|^3}\rho^{-L}, \quad \lambda \in \{\lambda \in \mathbb{R} : |\lambda| \ge 1+\epsilon, \epsilon > 0\}, \quad L \ge 1$$

and

$$1 < \rho < \min\{|\lambda_j| : |\lambda_j| > 1\}$$

Block-Toeplitz determinants and the RH problem

- It turns out that

$$V_-(z) = \sigma_3 U_-(z)^{-1} \sigma_3$$
$$V_+(z) = \sigma_3 U_+(z)^{-1} \sigma_3 (\lambda^2 - 1), \quad \lambda \neq \pm 1$$

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where

$$Q(z) = \begin{pmatrix} g(z) & -g(z) \\ i & i \end{pmatrix}, \quad \Lambda = i\begin{pmatrix} \lambda + 1 & 0 \\ 0 & \lambda - 1 \end{pmatrix}$$

Block-Toeplitz determinants and the RH problem

- Let us define

$$S(z) = U_{-}(z)Q(z)\Lambda^{-1}, \qquad |z| \geq 1,$$
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$$S(z) = U_-(z)Q(z)\Lambda^{-1}, \qquad |z| \geq 1,$$
$$S(z) = U_+(z)^{-1}Q(z), \qquad |z| \leq 1.$$

- S(z) is the unique solution of the Riemann-Hilbert problem

$$S_+(z) = S_-(z)\sigma_1, \qquad z \in \Sigma_i, \quad i = 1, \ldots, n$$
$$S_+(z) = S_-(z)\Lambda\sigma_1\Lambda^{-1} \qquad z \in \Sigma_i, \quad i = n+1, \ldots, 2n$$
$$\lim_{z\to\infty} S(z) = \lim_{z\to\infty} Q(z)\Lambda^{-1}.$$

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1. to solve the previous RH problem for $S(z)$ in terms of θ functions, and thus obtain the matrices $U_{\pm}(z)$ and $V_{\pm}(z)$;
2. to insert $U_{\pm}$ and $V_{\pm}(z)$ into Widom's formula

$$\frac{d}{d\lambda} \log D_L(\lambda) = -\frac{2\lambda}{1-\lambda^2} L$$
$$+\frac{1}{2\pi}\int_{\mathbb{S}^1} \operatorname{tr}\left[\left(U'_+(z)U_+^{-1}(z) + V_+^{-1}(z)V'_+(z)\right)\Phi^{-1}(z)\right] dz;$$

In order to compute the entropy of entanglement we need:

1. to solve the previous RH problem for $S(z)$ in terms of θ functions, and thus obtain the matrices $U_\pm(z)$ and $V_\pm(z)$;
2. to insert $U_\pm$ and $V_\pm(z)$ into Widom's formula

$$\frac{d}{d\lambda} \log D_L(\lambda) = -\frac{2\lambda}{1-\lambda^2} L + \frac{1}{2\pi} \int_{\mathbb{S}^1} \operatorname{tr}\left[\left(U_+'(z)U_+^{-1}(z) + V_+^{-1}(z)V_+'(z)\right)\Phi^{-1}(z)\right] dz;$$

3. to integrate

$$S(\rho_A) = \lim_{\epsilon\to 0^+}\left[\lim_{L\to\infty}\frac{1}{4\pi i}\oint_{\Gamma(\epsilon)} e(1+\epsilon,\lambda) \times \frac{d}{d\lambda}\log\left(D_L(\lambda)(\lambda^2-1)^{-L}\right) d\lambda\right].$$

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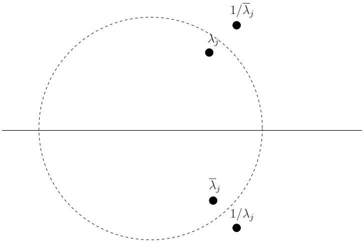

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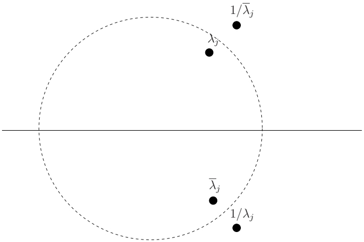

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$$\theta(\vec{s}) := \sum_{\vec{n} \in \mathbb{Z}^g} e^{i\pi \vec{n} \cdot \Pi \vec{n} - 2i\pi \vec{s} \cdot \vec{n}}.$$

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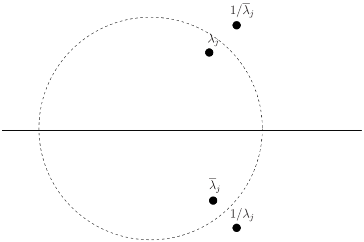

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  becomes degenerate.
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AR Its, F Mezzadri and MY Mo. Entanglement entropy in quantum spin chains with finite range interaction. arXiv:0708.0161v1.