Statistics of the number of zero crossings : from random polynomials to diffusion equation

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Phys. Rev. Lett. **99**, 060603 (2007), arXiv:0705.2648

Real Kac's polynomials

$$K_n(x) = \sum_{i=0}^{n-1} a_i x^i$$

$a_i \equiv$ independent Gaussian random variables, $\mathbb{E}\, a_i = 0$

Introduction : Real random polynomials

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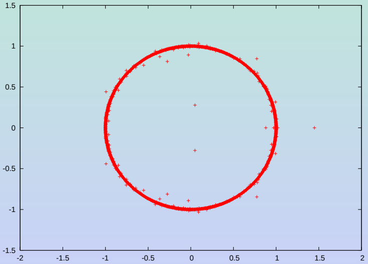
Complex roots



Real Kac's polynomials

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Real roots

$\mathcal{N}_n \equiv$ expected number of roots on the real axis (**M.Kac '43**)

$$\mathcal{N}_n \sim \frac{2}{\pi} \log n$$

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$$\mathcal{N}_n \sim \frac{2}{\pi} \log n$$

See also : Erdos, Maslova, Edelman, Kostlan, Bleher, Fyodorov...

Introduction : Real roots of Kac's polynomials

JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 15, Number 4, Pages 857–892
S 0894-0347(02)00386-7
Article electronically published on May 16, 2002

RANDOM POLYNOMIALS HAVING FEW OR NO REAL ZEROS

AMIR DEMBO, BJORN POONEN, QI-MAN SHAO, AND OFER ZEITOUNI

$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0,1]$

$$q_0(n) \propto n^{-\gamma}$$

$$\text{with} \quad \gamma = 0.19(1) \quad \text{(Numerics)}$$

Persistence probability $p_0(t)$

- $X(t) \equiv$ stochastic random variable evolving in time t, $\mathbb{E}\, X(t) = 0$
- Persistence probability

$$p_0(t) \equiv \text{Proba. that } X \text{ has not changed sign up to time } t$$

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Persistence in physical systems

- coarsening spin field at low T
- reaction-diffusion
- height of a fluctuating interface
- diffusion field
- ...

$$p_0(t) \propto t^{-\theta_p}$$

Motivations : persistence for the diffusion equation

Diffusion equation with random initial conditions

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\mathbb{E}\,\phi(x,0)\phi(x',0) = \delta^d(x-x')$$

Single length scale
$\ell(t) \propto t^{1/2}$

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Persistence $p_0(t,L)$ for a d-dim. system of linear size L

$p_0(t,L) \equiv$ Proba. that $\phi(x,t)$ has not changed sign up to t

S.N. Majumdar, C. Sire, A.J. Bray and Cornell, PRL'96

B. Derrida, V. Hakim and R. Zeitak, PRL'96

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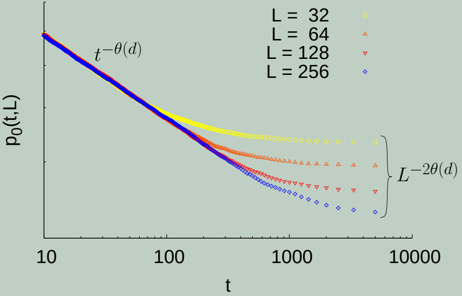

Motivations : persistence for the diffusion equation

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Persistence $p_0(t,L)$ for a d-dim. system of linear size L

$p_0(t,L) \equiv$ Proba. that $\phi(x,t)$ has not changed sign up to t

$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2)$$

$$\begin{aligned} \theta(1) &= 0.1207 \\ \theta(2) &= 0.1875 \quad , \quad \text{Numerics} \end{aligned}$$

Purpose : a link between random polynomials & diffusion equation

Generalized Kac's polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i \, i^{\frac{d-2}{4}} x^i$$

$a_i \equiv$ independent Gaussian random variables, $\mathbb{E}\, a_i = 0$

Proba. of no real root

$$q_0(n) \propto n^{-b(d)}$$

Persistence of diffusion

$$p_0(t, L) \propto L^{-2\theta(d)}$$

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$$b(d) = \theta(d)$$

Outline

1. Mapping to a Gaussian Stationnary Process (GSP)
   - The case of diffusion equation
   - The case of random polynomials
   - Numerical check
2. Probability of k-zero crossings
   - Generalization to k zero crossings for diffusion and polynomials
   - Large deviation function
   - Generating function
3. Conclusion

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Persistence of diffusion equation

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\mathbb{E}\, \phi(x,0)\phi(x',0) = \delta^d(x - x')$$

$$\phi(x,t) = \int_{|y|<L} d^d y G(x-y,t)\phi(y,0)$$

$$G(x,t) = (4\pi t)^{-\frac{d}{2}} \exp\left(-x^2/4t\right)$$

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Mapping of $\phi(x,t)$ to a Gaussian stationnary process

1. Normalized process $X(t) = \frac{\phi(x,t)}{(\mathbb{E}\phi(x,t)^2)1/2}$

$$\mathbb{E}X(t)X(t') \sim \begin{cases} \left(4\frac{tt'}{(t+t')^2}\right)^{\frac{d}{4}}, & t,t' \ll L^2 \\ 1, & t,t' \gg L^2 \end{cases}$$

2. New time variable $T = \log t$

$$\mathbb{E}X(T)X(T') = \left[\cosh((T-T')/2)\right]^{-d/2}$$

Persistence for a Gaussian stationnary process (GSP)

- $X(T)$ is a GSP with correlations

$$\begin{aligned} \mathbb{E}X(T)X(T') &= a(T-T') \\ a(T) &= (\cosh(T/2))^{-d/2} \end{aligned}$$

- Persistence probability $\mathcal{P}_0(T)$
- Slepian's Lemma

For $\quad T \gg 1 \quad a(T) \propto \exp\left(-\frac{d}{2}T\right) \Rightarrow \mathcal{P}_0(T) \propto \exp\left(-\theta(d)T\right)$

- Reverting back to $t = \exp(T)$

$$p_0(t,L) \sim t^{-\theta(d)} \quad 1 \ll t \ll L^2$$

Persistence of diffusion equation

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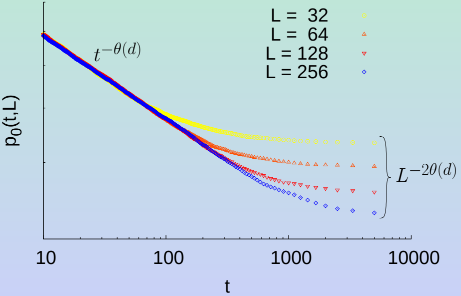

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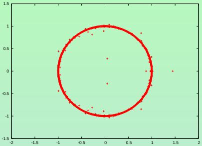
$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2) \qquad h(u) \sim \begin{cases} u \sim u^{-\theta(d)} \,, & u \ll 1 \\ u \sim c^{\mathrm{st}} \,, & u \gg 1 \end{cases}$$

Outline

Real roots of generalized Kac's polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$



Averaged density of real roots

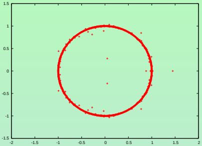
$$\rho_n(x) \;=\; \mathbb{E}[|K_n'(x)|\,\delta(K_n(x))]$$

Real roots concentrate around $x = \pm 1$

$$\rho_n(\pm 1) \sim A_d n \qquad A_d = \frac{2\sqrt{d/(d+4)}}{\pi(d+2)}$$

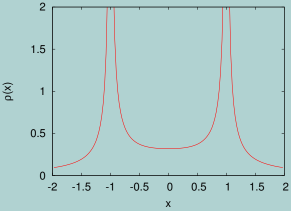
Real roots of generalized Kac's polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$



Averaged density of real roots for $n \to \infty$

$$\rho_\infty(x) = \frac{\left(\mathrm{Li}_{-1-d/2}(x^2)(1+\mathrm{Li}_{1-d/2}(x^2)) - \mathrm{Li}^2_{-d/2}(x^2)\right)^{\frac{1}{2}}}{\pi|x|(1+\mathrm{Li}_{1-d/2}(x^2))}$$



Real roots of generalized Kac's polynomials

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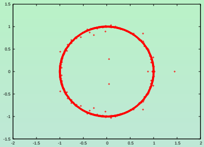
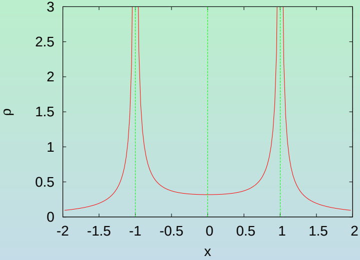


Expected number of real roots in $[0, 1]$: Kac-Rice formula

$$\mathbb{E}N_n[0,1] = \int_0^1 \rho_n(x)\, dx \sim \frac{1}{2\pi}\sqrt{\frac{d}{2}} \log n$$

Probability of no real root for $K_n(x)$



Dembo *et al.* **'02**

Statistical independance of $K_n(x)$ in the 4 sub-intervals

$\Longrightarrow$ Focus on the interval $[0, 1]$

$$P_0(x, n) \equiv \text{Proba. that } K_n(x) \text{ has no real root in } [0, x]$$

Probability of no real root for $K_n(x)$

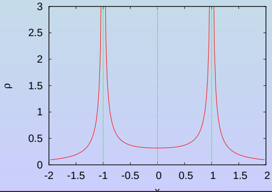
- Two-point correlator

$$C_n(x,y) = \mathbb{E}\, K_n(x) K_n(y) = \sum_{i=0}^{n-1} i^{(d-2)/2} (xy)^i$$

- Normalization

$$\mathcal{C}_n(x,y) = \frac{C_n(x,y)}{(C_n(x,x))^{\frac{1}{2}} (C_n(y,y))^{\frac{1}{2}}}$$

- Change of variable



$$x = 1 - \frac{1}{t} \quad , \quad t \gg 1$$

Probability of no real root for $K_n(x)$

Normalized correlator in the scaling limit

- Scaling limit

$$t \gg 1 \quad , \quad n \gg 1 \quad \text{keeping} \quad \tilde{t} = \frac{t}{n} \quad \text{fixed}$$

- $\mathcal{C}_n(t,t') \to \mathcal{C}(\tilde{t},\tilde{t}')$ with the asymptotic behaviors

$$\mathcal{C}(\tilde{t},\tilde{t}') \sim \begin{cases} \left(4\frac{\tilde{t}\tilde{t}'}{(\tilde{t}+\tilde{t}')^2}\right)^{\frac{d}{4}} \;, & \tilde{t},\tilde{t}' \ll 1 \\ 1 \,, & \tilde{t},\tilde{t}' \gg 1 \end{cases}$$

Persistence of diffusion equation (reminder)

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\mathbb{E}\,\phi(x,0)\phi(x',0) = \delta^d(x-x')$$

$$\phi(x,t) = \int d^d y G(x-y)\phi(y,0)$$
$$G(x,t) = (4\pi t)^{-\frac{d}{2}} \exp\left(-x^2/4t\right)$$

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2. Persistence probability $p_0(t,L)$

$$p_0(t,L) \propto L^{-2\theta(d)} h(t/L^2)$$

Probability of no real root for $K_n(x)$

$$\mathcal{C}(\tilde{t},\tilde{t}') \sim \begin{cases} \left(4\frac{\tilde{t}\tilde{t}'}{(\tilde{t}+\tilde{t}')^2}\right)^{\frac{d}{4}} \,, & \tilde{t},\tilde{t}' \ll 1 \\ 1 \,, & \tilde{t},\tilde{t}' \gg 1 \end{cases}$$

$P_0(x,n) \equiv$ Proba. that $K_n(x)$ has no real root in $[0,x]$

Scaling form for $P_0(x,n)$

$$P_0(x,n) \propto n^{-\theta(d)}\tilde{h}(n(1-x))$$

$$\tilde{h}(u) \sim \begin{cases} c^{st} \,, & u \ll 1 \\ u^{\theta(d)} \,, & u \gg 1 \end{cases}$$

Probability of no real root for $K_n(x)$

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$q_0(n) \equiv$ Probability that $K_n(x)$ has no real root in $[0, 1]$

$$q_0(n) = P(1, n) \sim n^{-\theta(d)}$$

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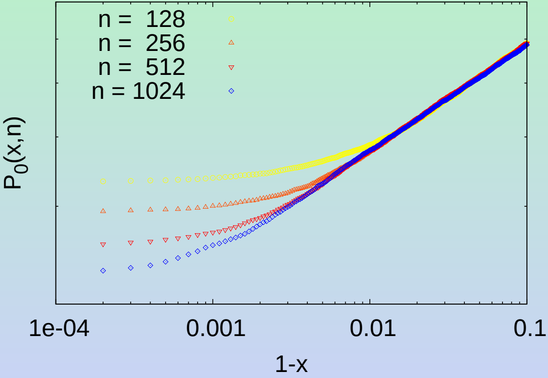
Numerical check of the scaling form

Numerical computation of $P_0(x,n)$ for $d = 2$



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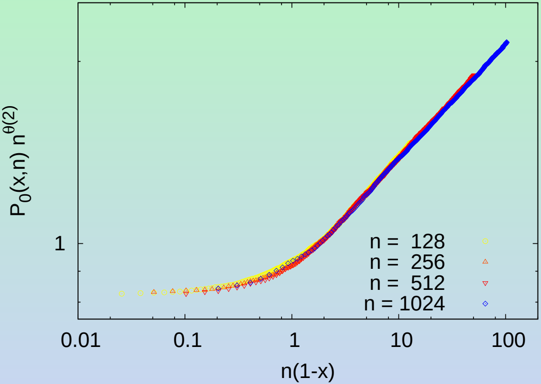
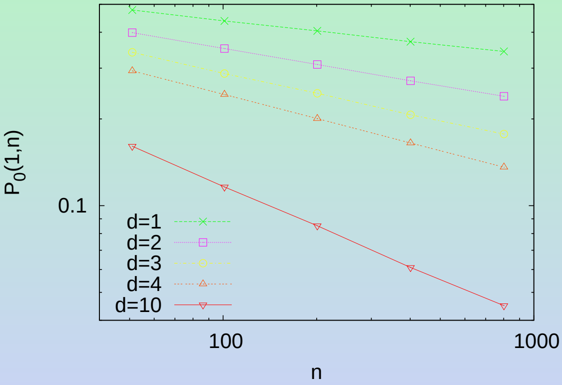


$$P_0(x,n) \propto n^{-\theta(d)} \tilde{h}(n(1-x))$$

Numerical check of the exponent

Numerical computation of $P(1,n) \propto n^{-\theta(d)}$



In agreement with the exponent $\theta(d)$ found for diffusion

Outline

Generalization to k zero crossings

- Real polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$

$q_k(n) \equiv$ Proba. that $K_n(x)$ has exactly k real roots

- Diffusion equation

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\mathbb{E}\phi(x,0)\phi(x',0) = \delta^d(x-x')$$

$p_k(t,L) \equiv$ Proba. that $\phi(x,t)$ crosses zero k times up to t

Outline

Scaling form for GSP : large deviation function

$X(T)$ is a GSP with $\mathbb{E}X(T)X(T') = [\mathrm{sech}((T-T')/2)]^{d/2}$

$\mathcal{P}_k(T) \equiv$ Proba. that $X(T)$ crosses 0 exactly k times up to T

$$\log \mathcal{P}_k(T) = -T\varphi\left(\frac{k}{\rho T}\right)$$

$\varphi(x)$ is a large deviation function

Outline

- Generating function **S.N. Majumdar, A.J. Bray, PRL'98**

$$\hat{\mathcal{P}}(z,T) = \sum_{k=0}^{\infty} z^k \mathcal{P}(k,T) \sim \exp(-\hat{\theta}(z)T)$$

where $\hat{\theta}(z)$ depends continuously on z

Generating function

- Generating function **S.N. Majumdar, A.J. Bray, PRL'98**

$$\hat{\mathcal{P}}(z,T) = \sum_{k=0}^{\infty} z^k \mathcal{P}(k,T) \sim \exp(-\hat{\theta}(z)T)$$

where $\hat{\theta}(z)$ depends continuously on z

- Assuming the proposed scaling form for $\mathcal{P}(k,T)$

$$\begin{aligned}\hat{\mathcal{P}}(z,T) &= \sum_{k=0}^{\infty} z^k \exp\left(-T\varphi\left(\frac{k}{\rho T}\right)\right) \\ &\propto \exp\left(-\mathrm{Min}_{x>0}\left[\varphi(x) - \rho x \log z\right]\right)\end{aligned}$$

Generating function

- Generating function **S.N. Majumdar, A.J. Bray, PRL'98**

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- Inverting the Legendre transform

$$\varphi(x) = \mathrm{Max}_{0\leq z\leq 2}[\rho x \log z + \hat{\theta}(z)]$$

Application to the integrated Brownian motion

- Integrated Brownian motion (“Random acceleration process”)

$$\mathcal{W}(t) \equiv \text{Brownian motion}$$
$$\mathcal{X}(t) = \int_0^t \mathcal{W}(s) ds$$

- Generating function **T.W. Burkhardt ’01**

$$\hat{\mathcal{P}}(z,T) = \sum_{k=0}^{\infty} z^k \mathcal{P}(k,T) \sim \exp(-\hat{\theta}(z)T)$$
$$\hat{\theta}(z) = \frac{1}{4}\left(1 - \frac{6}{\pi}\mathrm{ArcSin}\left(\frac{z}{2}\right)\right) \quad , \quad 0 \leq z \leq 2$$

Application to the integrated Brownian motion

- Integrated Brownian motion ("Random acceleration process")

$$\mathcal{W}(t) \equiv \text{Brownian motion}$$
$$\mathcal{X}(t) = \int_0^t \mathcal{W}(s)ds$$

- Exact result

$$\log \mathcal{P}_k(T) = -T\varphi\left(\frac{k}{\rho T}\right)\,,\ \rho = \frac{\sqrt{3}}{2\pi}$$
$$\varphi(x) = \frac{\sqrt{3}}{2\pi}x\log\left(\frac{2x}{\sqrt{x^2+3}}\right) + \frac{1}{4}\left(1 - \frac{6}{\pi}\mathrm{ArcSin}\left(\frac{x}{\sqrt{x^2+3}}\right)\right)$$

Probability of k-zero crossings for diffusion equation

$$\partial_t \phi(x,t) = \nabla^2 \phi(x,t)$$
$$\mathbb{E}\,\phi(x,0)\phi(x',0) = \delta^d(x-x')$$

$p_k(t,L) \equiv$ Proba. that $\phi(x,t)$ crosses zero k times up to t

$$p_k(t,L) \sim t^{-\varphi\left(\frac{k}{\log t}\right)} \quad , \quad t \ll L^2$$

Application to random polynomials

- Large deviation function for random polynomials

$$K_n(x) = a_0 + \sum_{i=1}^{n-1} a_i i^{(d-2)/4} x^i$$

$q_k(n) \equiv$ Proba. that $K_n(x)$ has exactly k real roots in $[0, 1]$

$$q_k(n) \propto n^{-\chi\left(\frac{k}{\log n}\right)}$$

Conclusion

- A link between diffusion equation and random polynomials

Proba. of no real root

$$q_0(n) \propto n^{-b(d)}$$

Persistence of diffusion

$$p_0(t, L) \propto L^{-2\theta(d)}$$

$$b(d) = \theta(d)$$

Conclusion

- $P_0(x,n) \equiv$ Proba. that $K_n(x)$ has no real root in $[0,x]$

$$P_0(x,n) \propto n^{-\theta(d)} \tilde{h}(n(1-x))$$

- $q_k(n) \equiv$ Proba. that $K_n(x)$ has exactly k real roots in $[0,1]$

$$q_k(n) \propto n^{-\chi\left(\frac{k}{\log n}\right)}$$

Perspectives

- Extension to other class of polynomials

$$W_n(x) = \sum_{i=0}^{n} \frac{a_i}{\sqrt{i\,!}}\, x^i$$

$$B_n(x) = \sum_{i=0}^{n} a_i \sqrt{\binom{n}{i}} x^i$$

- Extension to real eigenvalues of real random matrices
  i.e. Ginibre's matrices (c.f. **Akemann, Kanzieper)**

A heuristic argument

- Diffusion equation

$$\begin{aligned}
\phi(x=0,t) &= (4\pi t)^{-d/2} \int_{0<|\mathbf{x}|<L} d^d\mathbf{x}\ \exp\left(-\frac{\mathbf{x}^2}{4t}\right) \phi(\mathbf{x},0) \\
&= \frac{S_d^{1/2}}{(4\pi t)^{d/2}} \int_0^L dr\ r^{\frac{1}{2}(d-1)} e^{-\frac{r^2}{4t}}\ \Psi(r) \\
\Psi(r) &= S_d^{-1/2} r^{-\frac{1}{2}(d-1)} \lim_{\Delta r \to 0} \frac{1}{\Delta r} \int_{r<|\mathbf{x}|<r+\Delta r} d^d\mathbf{x}\ \phi(\mathbf{x},0) \\
\mathbb{E}\Psi(r)\Psi(r') &= \delta(r-r')
\end{aligned}$$

A heuristic argument

- Diffusion equation

$$\phi(x=0,t) \propto \int_0^{L^2} du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}}\, \tilde{\Psi}(u)$$

$$\mathbb{E}\, \tilde{\Psi}(u)\tilde{\Psi}(u') = \delta(u-u')$$

A heuristic argument

- Diffusion equation

$$\begin{aligned}\phi(x=0,t) &\propto \int_0^{L^2} du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}}\, \tilde{\Psi}(u) \\ \mathbb{E}\,\tilde{\Psi}(u)\tilde{\Psi}(u') &= \delta(u-u')\end{aligned}$$

- Random polynomials : $K_n(x) = a_0 + \sum_{i=1}^{n} a_i i^{\frac{d-2}{4}} x^i$

$$\begin{aligned}K_n(1-1/t) &\sim a_0 + \sum_{i=1}^{n} i^{\frac{d-2}{4}} e^{-\frac{i}{t}} a_i \\ &\sim \int_0^{n} du\, u^{\frac{d-2}{4}} e^{-\frac{u}{t}} a(u) \\ \mathbb{E}\, a(u)a(u') &= \delta(u-u')\end{aligned}$$