Correlation Kernels for Discrete Symplectic and Orthogonal Ensembles.

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§1. Correlation Kernels for Discrete Symplectic Ensembles 1

$\left(DSpE_N(\mathbb{Z}_{\geq 0}, \omega)\right)$

1.1 $DSpE_N(\mathbb{Z}_{\geq 0}, \omega)$: random point configuration $0 \leq x_1 < \dots < x_N \subset \mathbb{Z}_{\geq 0}$

$$\mathrm{Prob}(x_1, \dots, x_N) = \mathrm{Const} \cdot \prod_{i=1}^{N} \omega(x_i) \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 (x_i - x_j - 1)(x_i - x_j + 1)$$

where $\omega(x) > 0$ on $\mathbb{Z}_{\geq 0}$,

$\sum_{x \in \mathbb{Z}_{\geq 0}} x^j \omega(x)$ converges for all $j = 0, 1, 2, \dots$

1.2 Suppose there exists a 2×2 matrix valued kernel $K_{N4}(x, y)$ s.t. for a general finitely supported function η

$$\sum_{0 \leq x_1 < \dots < x_N \subset \mathbb{Z}_{\geq 0}} \mathrm{Prob}(x_1, \dots, x_N) \prod_{i=1}^{N} (1 + \eta(x_i)) = \sqrt{\det(1 + \eta K_{N4})}$$

where

- K_{N4} – operator associated to $K_{N4}(x, y)$
- η – operator of multiplication by $\eta(x)$

Then $K_{N4}(x, y)$ is called the <u>correlation kernel</u> of $DSpE_N(\mathbb{Z}_{\geq 0}, \omega)$

§2. Correlation Kernels for Discrete Orthogonal Ensembles ($DOE_N(\mathbb{Z}_{\geq 0}, \omega)$)

2.1 $DOE_N(\mathbb{Z}_{\geq 0}, \omega)$: random point configuration $x_1 < \ldots < x_{2N} \subset \mathbb{Z}_{\geq 0}$

$$\mathrm{Prob}(x_1, \ldots, x_{2N}) = \begin{cases} \text{const} \cdot \prod_{i=1}^{2N} W(x_i) \prod_{1 \leq i < j \leq 2N} (x_j - x_i), \text{ if } \underset{\text{even}}{\overset{x_1}{\uparrow}}, \underset{\text{odd}}{\overset{x_2}{\uparrow}}, \ldots \underset{\text{even}}{\overset{x_{2N-1}}{\uparrow}}, \underset{\text{odd}}{\overset{x_{2N}}{\uparrow}} \\ 0, \quad \text{otherwise} \end{cases}$$

2.2 Assumption: $W(x)W(x-1) = \omega(x)$ for $x \geq 1$, $W(0) = \omega(0)$.

2.3 Suppose there exists a 2×2 matrix valued kernel $K_{N1}(x,y)$ s.t. for a general finitely supported function η

$$\sum_{0 \leq x_1 < \ldots < x_{2N} \subset \mathbb{Z}_{\geq 0}} \mathrm{Prob}(x_1, \ldots, x_{2N}) \prod_{i=1}^{2N} (1 + \eta(x_i)) = \sqrt{\det(1 + \eta K_{N1})}$$

where

- K_{N1} – operator associated to $K_{N1}(x,y)$
- η – operator of multiplication by $\eta(x)$

Then $K_{N1}(x,y)$ is called the correlation kernel of $DOE_N(\mathbb{Z}_{\geq 0}, \omega)$

§3 Problem

Express $K_{N4}(x,y)$, $K_{N1}(x,y)$ in terms of orthogonal polynomials associated to ω

Applications

- Correlation functions for z-measures and Plancherel measures on Young diagrams with Jack parameters $\theta = 2, \frac{1}{2}$
- Parity respecting correlations for Orthogonal Ensembles of RMT.

The cases of special interest

- The Meixner weight

$$\omega_{\text{Meixner}}(x) = \frac{(\beta)_x}{x!} c^x, \quad x \in \mathbb{Z}_{\geq 0}, \quad \beta > 0, \quad 0 < c < 1.$$

- The Charlier weight

$$\omega_{\text{Charlier}}(x) = \frac{a^x}{x!}, \quad x \in \mathbb{Z}_{\geq 0}, \quad a > 0.$$

§4 Notation

- $\{P_n(\xi)\}_{n=0}^{+\infty}$ – orthogonal polynomials,

$$\sum_{x\in\mathbb{Z}_{\geq 0}} P_n(x)P_m(x)\,\omega(x)=0\,,\quad n\neq m$$

- $$\varphi_n(x)=\frac{P_n(x)}{\|P_n(.)\|}\,\omega^{1/2}(x)$$

$$\mathcal{H}=\mathrm{Span}\{\varphi_0,\varphi_1,\varphi_2,\dots\}$$

$$\mathcal{H}_N=\mathrm{Span}\{\varphi_0,\varphi_1,\dots,\varphi_{2N-1}\}$$

- $$(D_\pm\varphi)(x)=\sum_{y\in\mathbb{Z}_{\geq 0}} D_\pm(x,y)\,\varphi(y)$$

$$D_+(x,y)=\sqrt{\frac{\omega(x)}{\omega(x+1)}}\,\delta_{x,y}\,;\quad D_-(x,y)=\sqrt{\frac{\omega(x-1)}{\omega(x)}}\,\delta_{x,y}$$

- $$D:=D_+-D_-$$

- $$K_N=\sum_{j=0}^{2N-1}\varphi_j\otimes\varphi_j$$

- $$(\varepsilon\varphi)(2m) = -\sum_{k=m}^{+\infty} \sqrt{\frac{\omega(2m)}{\omega(2k+1)}}\, \frac{\omega(2m+1)\omega(2m+3)\dots\omega(2k+1)}{\omega(2m)\omega(2m+2)\dots\omega(2k)}\, \varphi(2k+1)$$

$$(\varepsilon\varphi)(2m+1) = \sum_{k=0}^{m} \sqrt{\frac{\omega(2k)}{\omega(2m+1)}}\, \frac{\omega(2k+1)\omega(2k+3)\dots\omega(2m+1)}{\omega(2k)\omega(2k+2)\dots\omega(2m)}\, \varphi(2k)$$

where $m = 0, 1, 2, \dots$

- **Assumptions**

$$\frac{\omega(x-1)}{\omega(x)} = \frac{d_1(x)}{d_2(x)}, \quad x \geq 1$$

for some polynomials d_1, d_2 s.t. $d_1(0)=0, d_2(0) \neq 0$,

$\deg d_1 \geq \deg d_2$

If $\deg d_1 = \deg d_2$ then $\lim_{x \to \infty} \frac{d_1(x)}{d_2(x)} > 1$.

This ensures the convergence of the series defining $(\varepsilon\varphi)(x)$ for any $\varphi \in \mathcal{H}$ and $x \in \mathbb{Z}_{\geq 0}$.

§5.

Theorem $$K_{N4} = \begin{bmatrix} D_+ S_{N4} & -D_+ S_{N4} D_- \\ S_{N4} & -S_{N4} D_- \end{bmatrix},$$

$$K_{N1} = \begin{bmatrix} S_{N1}\varepsilon & S_{N1} \\ \varepsilon S_{N1}\varepsilon - \varepsilon & \varepsilon S_{N1} \end{bmatrix}.$$

a

$$\omega(x) = \frac{(\beta)_x}{x!} c^x \quad (\text{the Meixner weight})$$

$$\boxed{\begin{aligned} S_{N4} &= \varepsilon K_N + \frac{\sqrt{2N(2N+\beta-1)}}{(1-c)\sqrt{c}} \, \varepsilon\psi_2 \otimes \varepsilon\psi_1 \\ S_{N1} &= D K_N + \frac{\sqrt{2N(2N+\beta-1)}}{(1-c)\sqrt{c}} \, \psi_2 \otimes \psi_1 \end{aligned}}$$

where $\psi_1(x) = \sqrt{2Nc}\,\dfrac{\varphi_{2N}(x)}{x+\beta} - \sqrt{2N+\beta-1}\,\dfrac{\varphi_{2N-1}(x)}{x+\beta}$

$$\psi_2(x) = \sqrt{2N+\beta-1}\,\frac{\varphi_{2N}(x)}{x+\beta-1} - \sqrt{2Nc}\,\frac{\varphi_{2N-1}(x)}{x+\beta-1}$$

b

$$\omega(x) = \frac{a^x}{x!} \quad (\text{the Charlier weight})$$

$$\boxed{\begin{aligned} S_{N4} &= \varepsilon K_N + \sqrt{2Na} \; \varepsilon\varphi_{2N} \otimes \varepsilon\varphi_{2N-1} \\ S_{N1} &= D K_N + \frac{\sqrt{2N}}{a} \; \varphi_{2N} \otimes \varphi_{2N-1} \end{aligned}}$$

C If $\frac{\omega(x-1)}{\omega(x)} = \frac{d_1(x)}{d_2(x)}$, for $x \geq 1$

for some polynomials d_1, d_2 s.t. $d_1(0)=0$, $d_2(0)\neq 0$, and $\deg d_1 \geq \deg d_2$ then

- $$\boxed{[D,K_N]K_N = \sum_{i=1}^{n} \tilde{\psi}_i \otimes \psi_i, \quad [\varepsilon,K_N]K_N = \sum_{i=1}^{n} \tilde{\eta}_i \otimes \eta_i}$$

where $\psi_1,\dots,\psi_n;\ \eta_1,\dots,\eta_n \in \mathcal{H}_N$.
$\tilde{\psi}_1,\dots,\tilde{\psi}_n;\ \tilde{\eta}_1,\dots,\tilde{\eta}_n \in \mathcal{H}_N^{\perp}$

- Assume in addition that the matrices

$$T_{ij} = \delta_{ij} + (\varepsilon\psi_i, \tilde{\psi}_j), \qquad 1 \leq i,j \leq n,$$
$$U_{ij} = \delta_{ij} + (D\eta_i, \tilde{\eta}_j)$$

are invertible. Then

$$\boxed{\begin{aligned} S_{N4} &= \varepsilon K_N - \sum_{i,j=1}^{n} (T^{-1})_{ij} (\varepsilon\tilde{\psi}_i)\otimes(K_N\varepsilon\psi_j) \\ S_{N1} &= DK_N - \sum_{i,j=1}^{n} (U^{-1})_{ij} (D\tilde{\eta}_i)\otimes(K_N D\eta_j) \end{aligned}}$$

Remark

Set

- $\dfrac{\omega(x-1)}{\omega(x)} = \dfrac{d_1(x)}{d_2(x)}$
- $d_2(x) = \text{const} \cdot (x-a_1)^{n_1} \cdots (x-a_\ell)^{n_\ell}$
- $n_\infty = $ the order of $\dfrac{\omega(x-1)}{\omega(x)}$ at ∞

Then

$$\boxed{n \le n_\infty + \sum_{i=1}^{\ell} n_{a_i}}$$

§ 6. z - measures on Young diagrams

- For $z, z' \in \mathbb{C}$, $\theta > 0$, $0 < \xi < 1$ define

$$M_{z,z',\theta,\xi}(\lambda) = (1-\xi)^{\frac{zz'}{\theta}} \cdot \xi^{|\lambda|} \frac{(z)_{\lambda,\theta}\,(z')_{\lambda,\theta}}{H(\lambda,\theta)\,H'(\lambda,\theta)},$$

where

$$(z)_{\lambda,\theta} = \prod_{(i,j)\in\lambda} \left(z + (j-i) - (i-1)\theta\right),$$

$$H(\lambda;\theta) = \prod_{(i,j)\in\lambda} \left((\lambda_i - j) + (\lambda'_j - i)\theta + 1\right),$$

$$H'(\lambda;\theta) = \prod_{(i,j)\in\lambda} \left((\lambda_i - j) + (\lambda'_j - i)\theta + \theta\right),$$

$|\lambda| = \#$ of boxes in λ, λ' denotes the transposed diagram.

- $$\sum_{\text{over all Young diagrams}} M_{z,z',\theta,\xi}(\lambda) = 1$$

Proposition.

Let $\mathbb{Y}(N)$ denote the set of diagrams λ with # of rows $\le N$.

Under the bijection between diagrams $\lambda \in \mathbb{Y}(N)$ and N-point configurations on $\mathbb{Z}_{\geq 0}$ defined by

$$\lambda \longleftrightarrow x_{N-i+1} = \lambda_i - 2i + 2N \quad (i=1,\dots,N)$$

the z-measure $M_{z,z',\theta,\xi}$ with parameters $\underline{z=2N}$, $\underline{\theta=2}$, $\underline{z'=2N+\beta-2}$ turns into

$$\mathrm{Prob}\{x_1,\dots,x_N\} = \mathrm{const}\cdot \prod_{i=1}^{N} \frac{(\beta)_{x_i}}{x_i!}\, \xi^{x_i} \prod_{1\le i<j\le N} (x_i-x_j)^2\left((x_i-x_j)^2-1\right)$$

This is the discrete symplectic ensemble with the Meixner weight $\omega(x) = \dfrac{(\beta)_x}{x!}\,\xi^x$.

Proposition.

Let $\mathbb{Y}'(2N)$ denote the set of Young diagrams with # of columns $\le 2N$. Under the Bijection between diagrams $\lambda \in \mathbb{Y}'(2N)$ and $2N$-point configurations on $\mathbb{Z}_{\geq 0}$ defined by

$$\lambda \longleftrightarrow x_{2N-i+1} = 2\lambda_i' - i + 2N \quad (i=1,\dots,2N)$$

the z-measure with parameters $\underline{z=-2N}$, $\underline{\theta=2}$, $\underline{z'=-2N-\beta}$ turns into

$$\mathrm{Prob}\{x_1,\dots,x_{2N}\} = \mathrm{const}\cdot \prod_{i=1}^{2N} \frac{[\beta]_{x_i}}{x_i!!}\, \xi^{\frac{x_i}{2}} \prod_{1\le i<j\le 2N} (x_j - x_i),$$

where

$$x_i!! = \begin{cases} 2\cdot 4\cdot \ldots \cdot x_i, & x_i \text{ is even}, \\ 1\cdot 3\cdot \ldots \cdot x_i, & x_i \text{ is odd}, \end{cases}$$

and

$$[\beta]_{x_i} = \begin{cases} (x_i+\beta-1)(x_i+\beta-3)\ldots(\beta+1), & x_i \text{ is even}, \\ (x_i+\beta-1)(x_i+\beta-3)\ldots\beta, & x_i \text{ is odd}. \end{cases}$$

This is a discrete orthogonal ensemble in the sense of our Definition (with the Meixner weight).

§7 Parity respecting correlations for the Laguerre Orthogonal Ensemble of RMT.

- The PDF for the Laguerre Orthogonal Ensemble is

$$\mathrm{const}\cdot\prod_{i=1}^{2N} e^{-\frac{z_i}{2}} z_i^{\frac{\alpha}{2}} \prod_{1\le j<k\le 2N} (z_j - z_k)$$

where $0 \le z_1 < z_2 < \ldots < z_{2N}$.

- Denote by $x_1, \ldots, x_N$ the odd labelled eigenvalues
  $y_1, \ldots, y_N$ the even labelled eigenvalues

- $$p^{(\alpha)}(x_1,\ldots,x_N; y_1,\ldots,y_N) := \mathrm{const} \prod_{i=1}^{2N} e^{-\frac{z_i}{2}} z_i^{\frac{\alpha}{2}} \prod_{1\le j<k\le 2N} (z_j - z_k)$$

- Set

$$S^{(\alpha)}_{(k_1,k_2)}(x_1,\ldots,x_{k_1}; y_1,\ldots,y_{k_2}) = \frac{N!}{(N-k_1)!}\cdot\frac{N!}{(N-k_2)!} \int_{(0,+\infty)^{k_1}}\int_{(0,+\infty)^{k_2}} p^{(\alpha)}(x_1,\ldots,x_N; y_1,\ldots,y_N) \prod_{l=k_1+1}^{N}\prod_{s=k_2+1}^{N} dx_l\, dy_s$$

This function describes parity respecting correlations of eigenvalues for the Laguerre Orthogonal Ensemble.

Theorem

$$\rho^{(d)}_{(k_1,k_2)}(x_1,\dots,x_{k_1};y_1,\dots,y_{k_2})$$

$$= \mathrm{Pf}\begin{bmatrix} \left[K_{N1}^{(d)}(x_j,x_l)\right]^{oo}_{j,l=1,\dots,k_1} & \left[K_{N1}^{(d)}(x_j,y_l)\right]^{oe}_{\substack{j=1,\dots,k_1\\ l=1,\dots,k_2}} \\ \left[K_{N1}^{(d)}(y_j,x_l)\right]^{eo}_{\substack{j=1,\dots,k_2\\ l=1,\dots,k_1}} & \left[K_{N1}^{(d)}(y_j,y_l)\right]^{ee}_{j,l=1,\dots,k_2} \end{bmatrix}$$

There are closed formulae for $\left[K_{N1}^{(d)}\right]^{oo}$, $\left[K_{N1}^{(d)}\right]^{oe}$, $\left[K_{N1}^{(d)}\right]^{eo}$, $\left[K_{N1}^{(d)}\right]^{ee}$.

For example,

$$\left[K_{N1}^{(d)}\right]^{oo} = \begin{bmatrix} \left[\varepsilon S_{N1}^{(d)}\right]^{oo} & \left[S_{N1}^{(d)}\right]^{oo} \\ \left[\varepsilon S_{N1}^{(d)}\varepsilon-\varepsilon\right]^{oo} & \left[S_{N1}^{(d)}\varepsilon\right]^{oo} \end{bmatrix}$$

where

- $\left[\varepsilon S_{N1}^{(d)}\right]^{oo} = K_N^{(d)} + \sqrt{2N(2N+d)}\left(\mathcal{E}^{o}\psi_2^{(d)}\right)\otimes\psi_1^{(d)}$
- $\left[\varepsilon S_{N1}^{(d)}\varepsilon-\varepsilon\right]^{oo} = \mathcal{E}^{o}K_N^{(d)} - \mathcal{E}^{o} + \sqrt{2N(2N+d)}\left(\mathcal{E}^{o}\psi_1^{(d)}\right)\otimes\left(\mathcal{E}^{o}\psi_2^{(d)}\right)$
- $\left[S_{N1}^{(d)}\right]^{oo} = \mathcal{D}K_N^{(d)} + \dfrac{\sqrt{2N(2N+d)}}{2}\ \psi_2^{(d)}\otimes\psi_1^{(d)}$
- $\left[S_{N1}^{(d)}\varepsilon\right]^{oo} = K_N^{(d)} + \sqrt{2N(2N+d)}\ \psi_1^{(d)}\otimes\left(\mathcal{E}^{o}\psi_2^{(d)}\right)$

Here $\left(\mathcal{E}^{o}f\right)(x) = -\frac{1}{2}\int_x^{+\infty} f(y)\,dy$, $\mathcal{D}$ is the operator of the differentiation

$\psi_1^{(d)}(x) = \sqrt{2N}\,\dfrac{\varphi_{2N}^{(d)}(x)}{x} - \sqrt{2N+d}\,\dfrac{\varphi_{2N-1}^{(d)}(x)}{x}$, $\psi_2^{(d)}(x) = \sqrt{2N+d}\,\dfrac{\varphi_{2N}^{(d)}(x)}{x} - \sqrt{2N}\,\dfrac{\varphi_{2N-1}^{(d)}(x)}{x}$,