One-parameter deformation of Wishart – Laguerre ensemble

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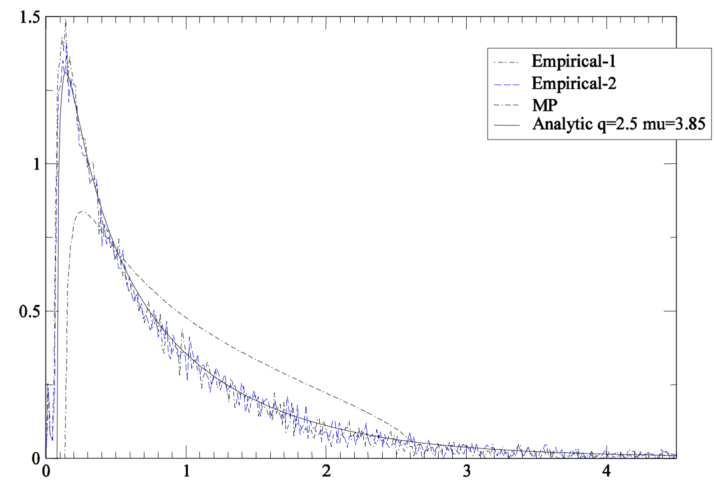
Motivations

- Empirical or experimental data (e.g. in finance or QCD) can be modelled by positive definite random matrices (covariance or chiral models).
- Often, their spectral properties follow a power law distribution (time series of assets – instantons and delocalization in QCD etc…).
- Few solvable model so far for positive definite random matrices with fat tails!
- A deformed Wishart-Laguerre ensemble (exactly solvable!) can serve the purpose….

Power Laws and Random Matrices

- Power-law density of levels: finance [covariance matrices]
- Power-law extreme eigenvalues



Spectral Density of covariance matrix of financial data [Biroli et al. 2007]

Wishart – Laguerre ensemble

$$P(\lambda_1,\ldots,\lambda_N) \propto \prod_{i=1}^{N} e^{-a\lambda_i} \lambda_i^{\nu} \prod_{j<k} |\lambda_j - \lambda_k|^2$$

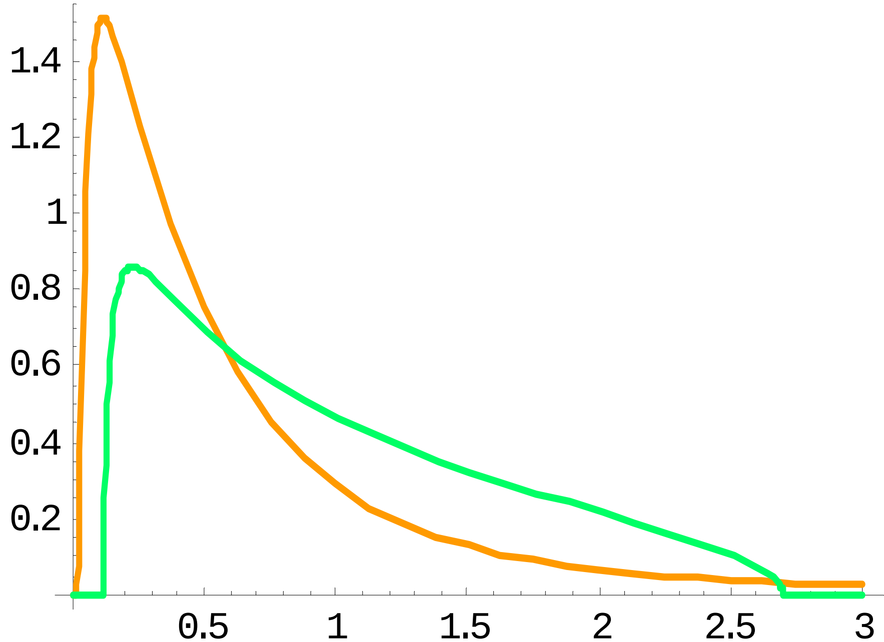
One-parameter deformation

$$P_{\gamma;(N,\nu)}(\lambda_1,\ldots,\lambda_N) \propto \left(1+\frac{a}{\gamma}\sum_{i=1}^{N}\lambda_i\right)^{-\gamma} \prod_{i=1}^{N} \lambda_i^{\nu} \prod_{j<k} |\lambda_j - \lambda_k|^2$$

$\gamma \to \infty$ Back to Wishart - Laguerre

Results

- Spectral Density
  - Finite N
  - Macroscopic
  - Microscopic



- Extreme eigenvalues
  - Finite N
  - Large N (in the chiral case)

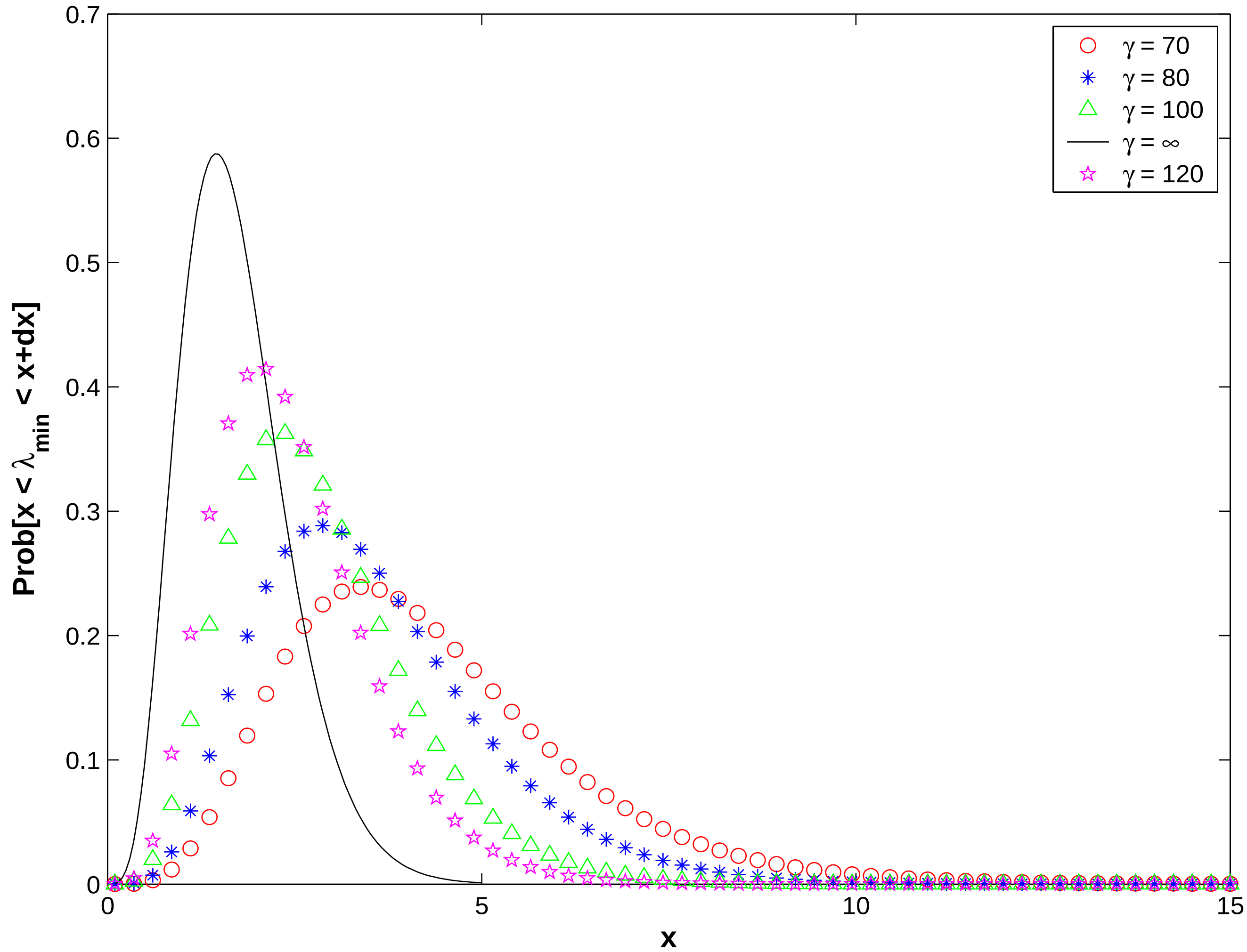
Gamma-transform Representation

$$\left(1+\frac{a}{\gamma}\,\mathrm{Tr}\;X\right)^{-\gamma}=\frac{1}{\Gamma(\gamma)}\int_0^\infty d\xi e^{-\xi}\xi^{\gamma-1}\exp\left(-\frac{a\xi}{\gamma}\,\mathrm{Tr}\;X\right)$$

[Bertuola *et al.* (PRE, 2004); Abul-Magd (PRE, 2005)]

This way, the power-law weight is mapped onto the classical Wishart-Laguerre, with one more integration needed.

Smallest Eigenvalue (Finite N)



$N = 5 \qquad \nu = 4$

Smallest Eigenvalue (Finite N)

ν positive integer for Wishart-Laguerre Unitary Ensemble: results by Dumitriu in terms of hypergeometric functions of a matrix argument.

For $\nu = 0$, simpler expression available

$$\text{Prob}[\, x < \lambda_{\min,\, N}^{(\nu=0)} < x + dx] = aN \exp\left(-aNx\right)$$

which gets modified in the deformed case as:

$$\text{Prob}[x < \lambda_{\min,N,\gamma}^{(\nu=0)} < x + dx] = \frac{aN(\gamma-1)}{\gamma}\left(1+\frac{aN}{\gamma}x\right)^{-\gamma}$$

Spectral Density (Macroscopic)

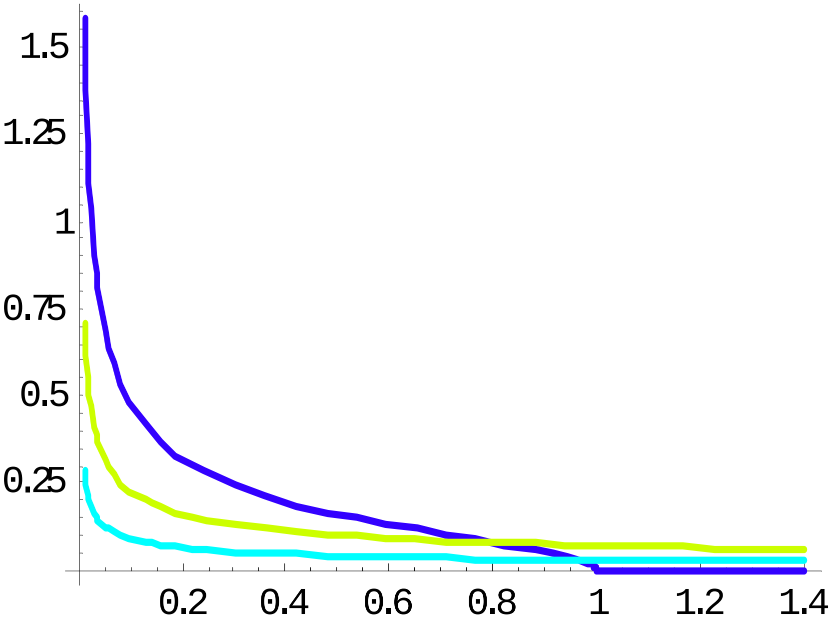
- Modification of the large N limit due to the constraint $\gamma > N^2$
- One can take $N \to \infty, \gamma \to \infty, \gamma - N^2 = \zeta > 0$

$$\lim_{N\to\infty} \left[\gamma N \rho_N (\gamma N \lambda)\right]^{\nu=0}_{\gamma - N^2 = \zeta} = \frac{\Gamma(\zeta + 1/2)}{\sqrt{\pi}\,\Gamma(\zeta)\Gamma(\zeta+2)} \left(\frac{4}{a}\right)^{\zeta} \times$$

$$\times \lambda^{-\zeta-1} \; {}_1F_1(\zeta + 1/2, \zeta + 2, -4/a\lambda)$$

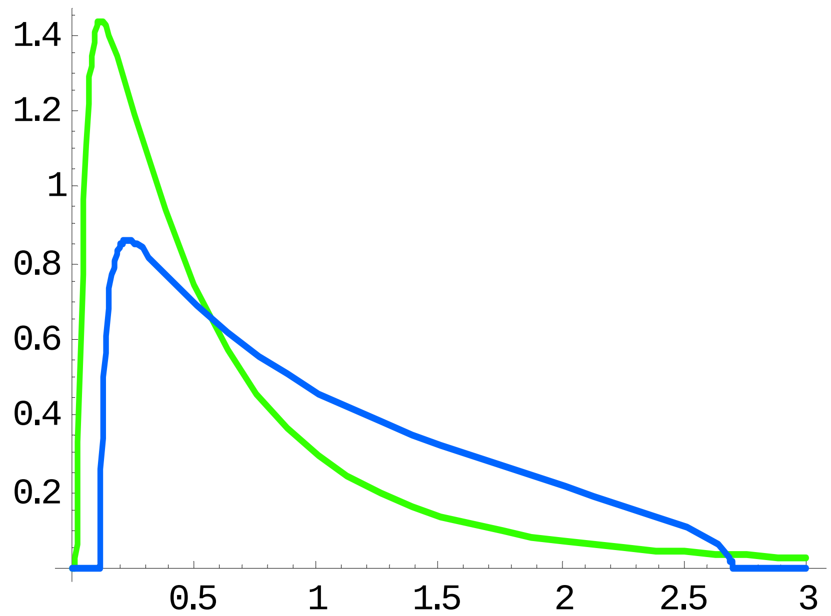
More complicated expression for $\nu > 0$

Spectral Density (Macroscopic)



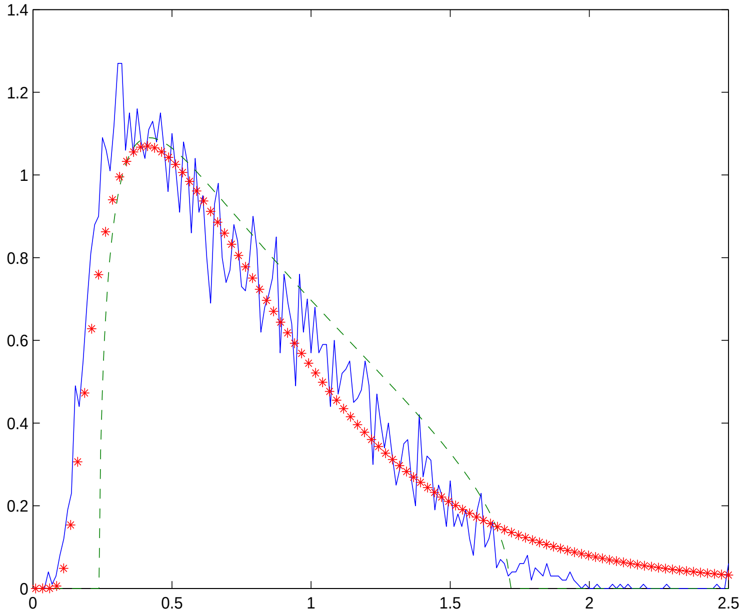
$\nu = 0$ Different values of ζ

[inverse square root divergence at the origin for all ζ, as in the original Marcenko-Pastur (blue)]

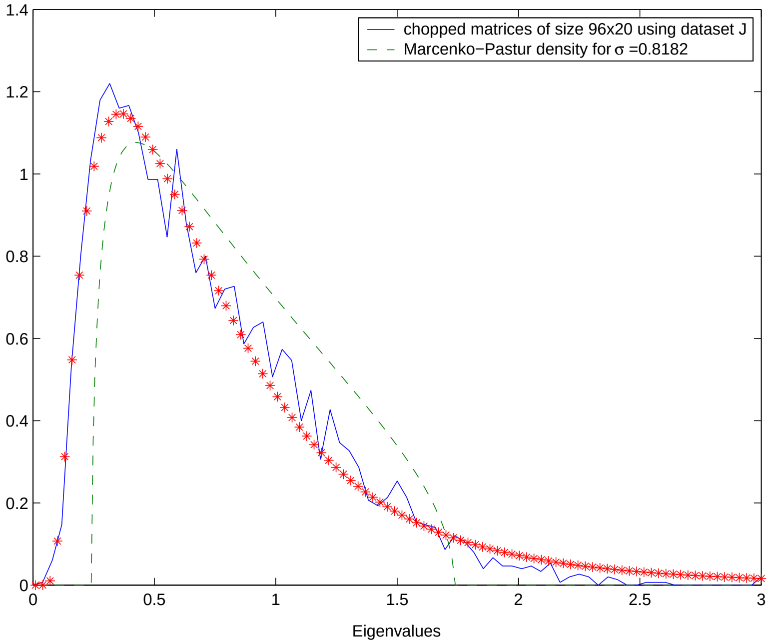


$\nu > 0$ Compared with Marcenko-Pastur (blue)

Comparison to Financial Data



$N = 10 \quad T = 48$



$N = 20 \quad T = 96$

Eigenvalue distribution for two ensembles of covariance matrices of financial data (N = number of assets; T = temporal series). Our macroscopic spectral density fits the curves reasonably well – much better than the raw Marcenko-Pastur – by adjusting the parameters ζ and a

References

[1] G. Akemann, P. Vivo and J. Fischmann
work in progress

[2] A. C. Bertuola, O. Bohigas, and M. P. Pato
Phys. Rev. E **70**, 065102 (2004)

[3] A. Y. Abul-Magd
Phys. Rev. E **71**, 066207 (2005)

[4] G. Biroli, J.-P. Bouchaud and M. Potters
arXiv: 0710.0802 (2007)