Superbosonization

(a new effective-field method for random matrix and disordered electron systems with local gauge symmetries)

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- Riemannian Symmetric Superspaces
- Hyperbolic Hubbard-Stratonovich Transformation
- Superbosonization

Riemannian Symmetric Superspaces

Symmetric spaces

Symmetric space :=

Globally symmetric Riemannian manifold

$$M = G/H\,, \quad \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}\,, \quad \mathfrak{p} \cong T_x M\,,$$

$$\mathfrak{g} = \mathrm{Lie}(G), \quad \mathfrak{h} = \mathrm{Lie}(H)$$

Spontaneous breaking of compact symmetries:
M as the space of the “order parameter”

Example: (ferro-)magnetic order, $M = \mathrm{SO}_3 / \mathrm{SO}_2$

Generalization: symmetric supermanifolds

What's a Riemannian symmetric superspace?

Data:

1) complex Lie superalgebra with Cartan involution:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = (\mathfrak{h}_0 \oplus \mathfrak{p}_0) \oplus (\mathfrak{h}_1 \oplus \mathfrak{p}_1)$$

2) symmetric space M with $T_x M \otimes \mathbb{C} \cong \mathfrak{p}_0$

Compact Lie group H with $\mathrm{Lie}(H) \otimes \mathbb{C} = \mathfrak{h}_0$

View M as base of principal bundle $G \to G/H = M$

Construct associated vector bundle $F = G \times_H \mathfrak{p}_1 \to M$

with fibre $F_x \cong \mathfrak{p}_1$

Riemannian symmetric superspaces: definition

Consider sections of $\wedge(F_x^*) \to M$ (superfunctions),
i.e., functions ψ on M with values $\psi(x) \in \wedge(F_x^*)$

Fact : the algebra of superfunctions $\mathcal{A} \equiv \Gamma(M, \wedge(F^*))$ carries an action of the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ by derivations.

Supermatrix model…

Definition : the supermanifold $(M, \mathcal{A})$ equipped with its $\mathfrak{g}$ –invariant (super–)geometry $B: \ \mathrm{Der}(\mathcal{A}) \times \mathrm{Der}(\mathcal{A}) \to \mathcal{A}$ is called a Riemannian symmetric superspace.

RSS arise as target spaces of supersymmetric nonlinear sigma models.

Example

Consider Grassmann manifold $M_1 = \mathrm{U}_{r+s}/(\mathrm{U}_r \times \mathrm{U}_s)$
of orthogonal decompositions $\mathbb{C}^{r+s} = V_1 \oplus V_1^{\perp} \quad (\dim V_1 = r)$

$\mathbb{C}^{p+q}$ with (pseudo–)Hermitian structure h of signature (p,q)
determines Grassmann manifold $M_0 = \mathrm{U}_{p,q}/(\mathrm{U}_p \times \mathrm{U}_q)$
of h–orthogonal decompositions $\mathbb{C}^{p+q} = V_0 \oplus V_0^{\perp} \quad (\dim V_0 = p)$

Symmetric space $M = M_0 \times M_1$

Tangent space: $T_x M \otimes \mathbb{C} = \mathrm{Hom}(V_0, V_0^{\perp}) \oplus \mathrm{Hom}(V_0^{\perp}, V_0)$
$\oplus\ \mathrm{Hom}(V_1, V_1^{\perp}) \oplus \mathrm{Hom}(V_1^{\perp}, V_1)$

Fibre of vector bundle: $F_x = \mathrm{Hom}(V_0, V_1^{\perp}) \oplus \mathrm{Hom}(V_1^{\perp}, V_0)$
$\oplus\, \mathrm{Hom}(V_1, V_0^{\perp}) \oplus \mathrm{Hom}(V_0^{\perp}, V_1)$

Example (continued)

$$\begin{aligned} V_0 &\cong \mathbb{C}^p \\ V_0^\perp &\cong \mathbb{C}^q \\ V_1 &\cong \mathbb{C}^r \\ V_1^\perp &\cong \mathbb{C}^s \end{aligned} \qquad \begin{pmatrix} \mathfrak{h}_0 & \mathfrak{p}_0 & \mathfrak{h}_1 & \mathfrak{p}_1 \\ \mathfrak{p}_0 & \mathfrak{h}_0 & \mathfrak{p}_1 & \mathfrak{h}_1 \\ \mathfrak{h}_1 & \mathfrak{p}_1 & \mathfrak{h}_0 & \mathfrak{p}_0 \\ \mathfrak{p}_1 & \mathfrak{h}_1 & \mathfrak{p}_0 & \mathfrak{h}_0 \end{pmatrix}$$

Lie superalgebra $\mathfrak{g} = (\mathfrak{h}_0 \oplus \mathfrak{p}_0) \oplus (\mathfrak{h}_1 \oplus \mathfrak{p}_1)$

$$= \mathfrak{gl}(\mathbb{C}^{p+q} \mid \mathbb{C}^{r+s}) = \mathfrak{gl}_{p+q|r+s}$$

Important:

Real form $T_x M$ of $\mathfrak{p}_0$ carries positive quadratic form.

Table: the Tenfold Way

this talk:

HS

SB

M_0	M_1	$\mathfrak{g}$	$\mathfrak{h}$
$\mathrm{U}_{p,q}/\mathrm{U}_p \times \mathrm{U}_q$	$\mathrm{U}_{r+s}/(\mathrm{U}_r \times \mathrm{U}_s)$	$\mathfrak{gl}_{p+q\|r+s}$	$\mathfrak{gl}_{p\|r} \oplus \mathfrak{gl}_{q\|s}$
$\mathrm{Sp}_{2m}(\mathbb{R})/\mathrm{U}_m$	$\mathrm{SO}_{2n}/\mathrm{U}_n$	$\mathfrak{osp}_{2m\|2n}$	$\mathfrak{gl}_{m\|n}$
$\mathrm{SO}^*_{2m}/\mathrm{U}_m$	$\mathrm{USp}_{2n}/\mathrm{U}_n$	$\mathfrak{osp}_{2m\|2n}$	$\mathfrak{gl}_{m\|n}$
$\mathrm{GL}_m(\mathbb{C})/\mathrm{U}_m$	U_n	$\mathfrak{gl}_{m\|n} \oplus \mathfrak{gl}_{m\|n}$	$\mathfrak{gl}_{m\|n}$
$\mathrm{O}_m(\mathbb{C})/\mathrm{O}_m$	USp_{2n}	$\mathfrak{osp}_{m\|2n} \oplus \mathfrak{osp}_{m\|2n}$	$\mathfrak{osp}_{m\|2n}$
$\mathrm{Sp}_{2m}(\mathbb{C})/\mathrm{USp}_{2m}$	O_n	$\mathfrak{osp}_{2m\|n} \oplus \mathfrak{osp}_{2m\|n}$	$\mathfrak{osp}_{2m\|n}$
$\mathrm{O}_{p,q}/(\mathrm{O}_p \times \mathrm{O}_q)$	$\mathrm{USp}_{2r+2s}/(\mathrm{USp}_{2r} \times \mathrm{USp}_{2s})$	$\mathfrak{osp}_{p+q\|2r+2s}$	$\mathfrak{osp}_{p\|2r} \oplus \mathfrak{osp}_{q\|2s}$
$\mathrm{Sp}_{2p,2q}/(\mathrm{USp}_{2p} \times \mathrm{USp}_{2q})$	$\mathrm{O}_{r+s}/(\mathrm{O}_r \times \mathrm{O}_s)$	$\mathfrak{osp}_{2p+2q\|r+s}$	$\mathfrak{osp}_{2p\|r} \oplus \mathfrak{osp}_{2q\|s}$
$\mathrm{GL}_m(\mathbb{R})/\mathrm{O}_m$	$\mathrm{U}_{2n}/\mathrm{USp}_{2n}$	$\mathfrak{gl}_{m\|2n}$	$\mathfrak{osp}_{m\|2n}$
$\mathrm{GL}_m(\mathbb{H})/\mathrm{USp}_{2m}$	$\mathrm{U}_n/\mathrm{O}_n$	$\mathfrak{gl}_{2m\|n}$	$\mathfrak{osp}_{2m\|n}$

M.R.Z., Riemannian symmetric superspaces and their origin in random matrix theory, J. Math. Phys. **37** (1996) 4986

Hyperbolic Hubbard-Stratonovich Transformation

- Joint work with Y. Fyodorov and Y. Wei (preprint available upon request)
- Fyodorov, J. Phys. Condensed Matter **17** (2005) S1915
- Wei & Fyodorov, J. Phys. A **40** (2007) 13587

Background & motivation

Consider real symmetric $N \times N$ matrices H

$$\left\langle \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\,HK} \right\rangle_{\mathrm{GOE}} = \int \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\,HK} d\mu(H) = \mathrm{e}^{-(b^2/2N)\,\mathrm{Tr}\,K^2}$$

Expectation of reciprocal of product of characteristic polynomials:

$$F(z) = \left\langle \prod_{j=1}^{p+q} \mathrm{Det}^{-1}(z_j - H) \right\rangle_{\mathrm{GOE}} \qquad (z_j \in \mathbb{C} \setminus \mathbb{R})$$

with $\mathrm{Im}\, z_j > 0$ for $j = 1, \ldots, p$ and $\mathrm{Im}\, z_j < 0$ for $j = p+1, \ldots, p+q$

Use $\mathrm{Det}^{-1}(z_j - H) = (\mathrm{i}\sqrt{\pi} s_j)^{-N} \int \exp \mathrm{i} s_j (\varphi, \varphi z_j - H\varphi) \, | d\varphi |$,

to obtain $\qquad s_j = \mathrm{sign}\, \mathrm{Im}(z_j)$

$$F(z) = \int \mathrm{e}^{\mathrm{i} \sum_{j=1}^{p+q} s_j z_j (\varphi_j, \varphi_j) - (b^2/2N) \mathrm{Tr}\, A(\varphi)^2} \prod_j (\mathrm{i}\sqrt{\pi} s_j)^{-N} \, | d\varphi_j |$$

Background & motivation (continued)

$$A(\varphi)_{ij} = \sum_{a=1}^{N} \varphi_{i,a}\, \varphi_{j,a}\, s_j$$

Note: $A^{\mathrm{t}} = sAs$ with $s = \mathrm{diag}(\mathrm{Id}_p\,, -\mathrm{Id}_q)$ is invariant under conjugation by $g^{-1} = sg^{\mathrm{t}}s \in \mathrm{O}_{p,q}$ ("hyperbolic symmetry", Wegner 1979)

Next step is Hubbard-Stratonovich decoupling:

$$C_0\ \mathrm{e}^{-\mathrm{Tr}\,A^2} = \int_D \mathrm{e}^{-\mathrm{Tr}\,R^2 - 2\mathrm{i}\,\mathrm{Tr}\,AR}\, |\,dR\,|$$

Problem: how is this done correctly?!

$A^{\mathrm{t}} = sAs$ forces $R^{\mathrm{t}} = sRs$ but $\mathrm{Tr}\,R^2 = \mathrm{Tr}\,RsR^{\mathrm{t}}s$ is of indefinite sign.

Domain of Schäfer – Wegner (1980) is not $\mathrm{O}_{p,q}$ – invariant.

Pruisken-Schäfer domain

$$Rv = \lambda v \ : \quad \begin{array}{ll} v^{t} s v > 0 & \text{space-like,} \quad \bullet \\ v^{t} s v < 0 & \text{time-like,} \quad \circ \end{array}$$

Every $\mathrm{O}_{p,q}$ – diagonalizable matrix R has p space – like and q time – like eigenvalues.

Encode ordering by motif, e.g., $\sigma(R) = \bullet\circ\bullet\circ\circ\bullet \quad (p = q = 3)$.
Associate with each motif σ a domain D_σ by closure.

Pruisken – Schäfer domain $D = \bigcup_\sigma D_\sigma$ is a union of $\binom{p+q}{p} = \binom{p+q}{q}$ domains. Each D_σ is $\mathrm{O}_{p,q}$ – invariant.
$D_\sigma \cap D_{\sigma'}$ for $\sigma \neq \sigma'$ has co – dimension 2.

Statement of result

Let $|dR| = \prod_{i\leq j} dR_{ij}$ (Lebesgue measure)

Theorem (FWZ). There exists some choice of cutoff function $R \mapsto \chi_\varepsilon(R)$ (converging pointwise to unity as $\varepsilon \to 0$) and a unique choice of sign function $\sigma \mapsto \mathrm{sgn}(\sigma) \in \{\pm 1\}$ and a number $C_{p,q}$ such that

$$C_{p,q} \lim_{\varepsilon\to 0} \sum_\sigma \mathrm{sgn}(\sigma) \int_{D_\sigma} \mathrm{e}^{-\mathrm{Tr}\, R^2 - 2\mathrm{i}\, \mathrm{Tr}\, AR} \chi_\varepsilon(R)\, |dR| \; = \; \mathrm{e}^{-\mathrm{Tr}\, A^2}$$

holds true for all matrices $A = sA^{\mathrm{t}} s$ with the property $As > 0$.

Remark. $\mathrm{sgn}(\sigma)$ is the parity of the number of transpositions $\bullet \leftrightarrow \circ$ needed to reduce σ to the extremal form $\sigma_0 = \bullet\bullet\ldots\bullet\circ\circ\ldots\circ$.

Corollary

Formulation in terms of eigenvalues: Let $R = g\lambda g^{-1}$ with $\lambda = \mathrm{diag}(\lambda_1, ..., \lambda_{p+q})$ and $g \in \mathrm{SO}_{p,q}$. Volume element $|dR| = J(\lambda)\,|d\lambda|\,dg$ where $|d\lambda| = \prod_{i=1}^{p+q} d\lambda_i$ and dg Haar measure for $\mathrm{SO}_{p,q}$. Jacobian: $J(\lambda) = \prod_{i<j} |\lambda_i - \lambda_j|$.

Corollary (FWZ). Define a sign-alternating Jacobian $J'(\lambda)$ by

$$J'(\lambda) = J(\lambda) \prod_{i=1}^{p} \prod_{j=p+1}^{p+q} \mathrm{sign}(\lambda_i - \lambda_j) = J(\lambda)\,\mathrm{sgn}(\sigma(\lambda)).$$

Then, if A satisfies the condition $sA^{\mathrm{t}} = As > 0$ we have

$$C_{p,q} \lim_{\varepsilon\to 0} \int_{\mathbb{R}^{p+q}} \left(\int_{\mathrm{SO}_{p,q}} \mathrm{e}^{-2\mathrm{i}\,\mathrm{Tr}\,A g \lambda g^{-1}} \chi_\varepsilon(g\lambda g^{-1})\,dg \right) \mathrm{e}^{-\mathrm{Tr}\,\lambda^2} J'(\lambda)\,|d\lambda| \;=\; \mathrm{e}^{-\mathrm{Tr}\,A^2}.$$

Idea of proof

Remark. $p = q = 1$: Fyodorov 2005

$p = 2,\ q = 1$: Wei, Fyodorov 2007

Consider

$$I(A) := \lim_{\varepsilon \to 0} \sum_{\sigma} \operatorname{sgn}(\sigma) \int_{D_\sigma} \mathrm{e}^{-\operatorname{Tr}(R+\mathrm{i}A)^2} \chi_\varepsilon(R+\mathrm{i}A)\, |dR| \,.$$

Show that

$$\left.\frac{d}{dt} I(A+t\dot{A})\right|_{t=0} = \lim_{\varepsilon \to 0} \mathrm{i} \sum_{\sigma} \operatorname{sgn}(\sigma) \int_{\partial D_\sigma} \mathrm{e}^{-\operatorname{Tr}(R+\mathrm{i}A)^2} \chi_\varepsilon(R+\mathrm{i}A)\, \iota_{\tau(\dot{A})}\, |dR|$$

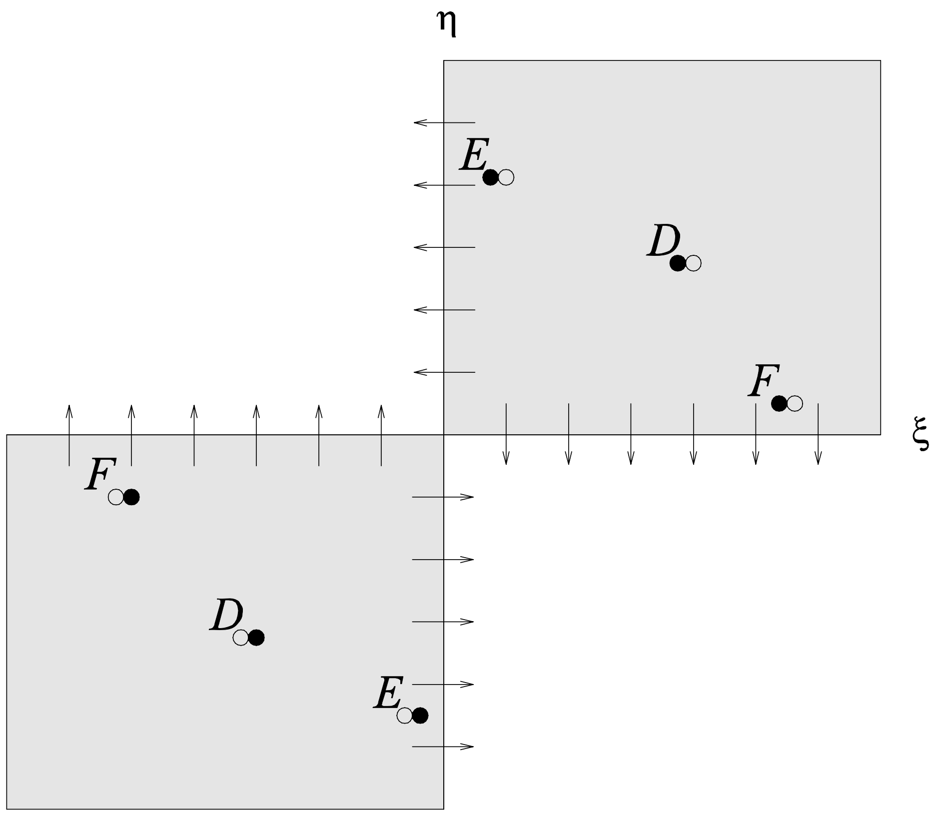
vanishes.

Two domains for p = q = 1

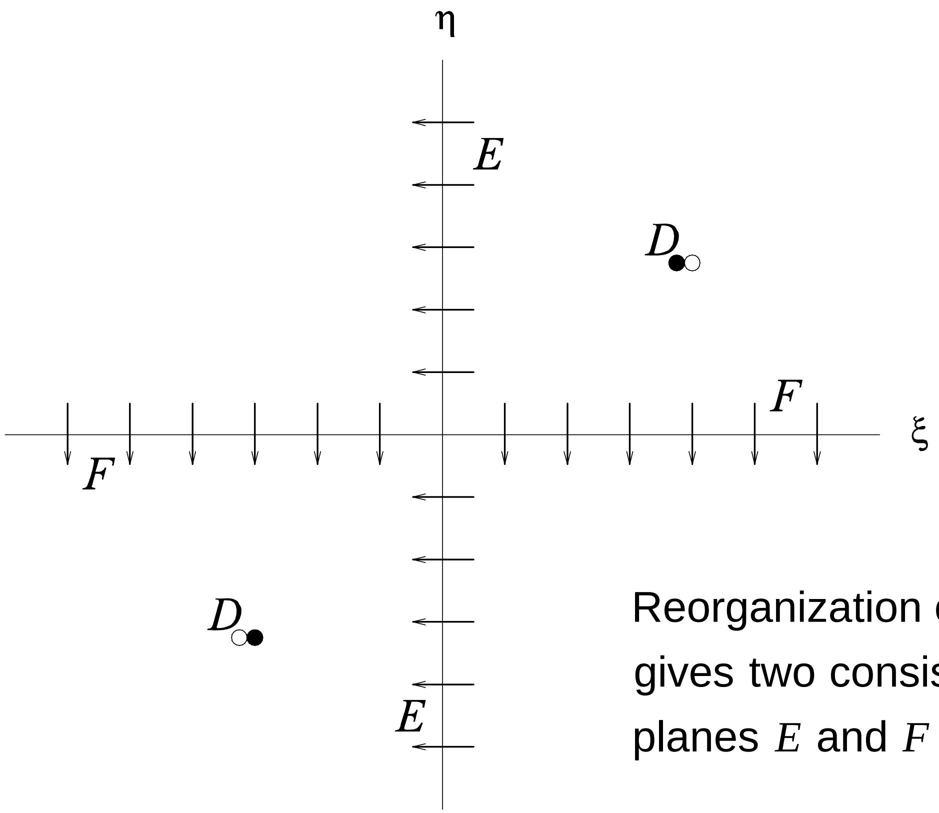
$$R = \begin{pmatrix} r_{11} & r_{12} \\ -r_{12} & r_{22} \end{pmatrix}$$

Light-cone coordinates:

$$\xi = (r_{11} - r_{22})/2 - r_{12}$$
$$\eta = (r_{11} - r_{22})/2 + r_{12}$$



Reorganization of boundary components



Reorganization of the boundary pieces gives two consistently oriented boundary planes E and F.

Superbosonization

- P. Littelmann, H.-J. Sommers, M.R.Z., Commun. Math. Phys. (in press)
- J.E. Bunder, K.B. Efetov, V.E. Kravtsov, O.M. Yevtushenko, M.R.Z.,
  J. Stat. Phys. 129 (2007) 809

Motivation: supersymmetry method

Central object of the theory is the characteristic function of the disorder distribution, $\Omega(K)=\int e^{\mathrm{i}\,\mathrm{Tr}\,HK}d\mu(H)$, where ($Z$ commuting, ζ anticommuting variables) $K_{ab}=\sum_{i=1}^{p} Z_{a,i}\,\tilde{Z}_{i,b}+\sum_{j=1}^{q}\zeta_{a,j}\,\tilde{\zeta}_{j,b}$

Generating function for spectral correlation functions:

$$\int_{\tilde{Z}=Z^*} D_{Z,\tilde{Z};\zeta,\tilde{\zeta}}\, f(Z,\tilde{Z},\zeta,\tilde{\zeta}) \equiv \int f \quad \text{where}$$

$$f(Z,\tilde{Z},\zeta,\tilde{\zeta}) = \Omega(K)\, e^{\mathrm{i}\sum_{i,a} Z_{a,i}E_i\tilde{Z}_{i,a}+\mathrm{i}\sum_{j,b}\zeta_{b,j}E_j^{\prime}\tilde{\zeta}_{j,b}}.$$

Assume $d\mu(H)$ invariant by some group G acting by conjugation $H\mapsto gHg^{-1}$. Then $f(Z,\tilde{Z},\zeta,\tilde{\zeta}) = f(gZ,\tilde{Z}g^{-1},g\zeta,\tilde{\zeta}g^{-1})$.

Special case: commuting variables only

Let $p = 1,\ q = 0$ and consider GL_N –invariant holomorphic function $f : \mathbb{C}^N \times (\mathbb{C}^N)^* \to \mathbb{C},\ f(z, \tilde{z}) = f(gz, \tilde{z}g^{-1}),\ g \in \mathrm{GL}_N$.

Fact (from invariant theory) : there exists a holomorphic function $F : \mathbb{C} \to \mathbb{C}$ such that $F(\tilde{z} \cdot z) = f(z, \tilde{z})$.

By push forward of the integral one has
$\int_{\mathbb{C}^N} f(z, z^*)\, d^{2N} z = c_N \int_{\mathbb{R}_+} F(r)\, r^{N-1} dr$ (if the integral exists).

generalization: see Fyodorov, Nucl. Phys. B **621** (2002) 643

Special case: Grassmann variables only

Let $F:\mathbb{C}\to\mathbb{C}$ be a holomorphic function. For a vector ψ of anticommuting variables $\psi_1,...,\psi_N$ consider $F(\overline{\psi}\cdot\psi)$.

Berezin integral $\int F(\overline{\psi}\cdot\psi)\, d\overline{\psi}\, d\psi \;:=$

$$= \frac{\partial^2}{\partial\psi_1\partial\overline{\psi}_1}\cdots\frac{\partial^2}{\partial\psi_N\partial\overline{\psi}_N}\, F(\overline{\psi}_1\psi_1+...+\overline{\psi}_N\psi_N)$$

$= F^{(N)}(0)$ (the $N-$th derivative at the origin)

$$= N!\oint_{\mathrm{U}(1)} F(z)\, z^{-N-1} dz/2\pi\,\mathrm{i}.$$

Kawamoto and Smit, Nucl. Phys. B **192** (1981) 100

A statement of result

Recall $f(Z,\tilde{Z},\zeta,\tilde{\zeta}) \coloneqq f(gZ,\tilde{Z}g^{-1},g\zeta,\tilde{\zeta}g^{-1})$ for $g \in G$.

Let $G = \mathrm{GL}_N$ or $G = \mathrm{O}_N$ or $G = \mathrm{Sp}_N$.

Superbosonization exploits this symmetry to make reduction.

Example : $G = \mathrm{O}_N$

$$f(Z,\tilde{Z},\zeta,\tilde{\zeta}) = F\begin{pmatrix} \tilde{Z}Z & \tilde{Z}\tilde{Z}^{\mathrm{t}} & \tilde{Z}\zeta & -\tilde{Z}\tilde{\zeta}^{\mathrm{t}} \\ Z^{\mathrm{t}}Z & Z^{\mathrm{t}}\tilde{Z}^{\mathrm{t}} & Z^{\mathrm{t}}\zeta & -Z^{\mathrm{t}}\tilde{\zeta}^{\mathrm{t}} \\ \tilde{\zeta}Z & \tilde{\zeta}\tilde{Z}^{\mathrm{t}} & \tilde{\zeta}\zeta & -\tilde{\zeta}\tilde{\zeta}^{\mathrm{t}} \\ \zeta^{\mathrm{t}}Z & \zeta^{\mathrm{t}}\tilde{Z}^{\mathrm{t}} & \zeta^{\mathrm{t}}\zeta & -\zeta\tilde{\zeta}^{\mathrm{t}} \end{pmatrix}$$

A statement of result (continued)

Theorem (LSZ). If $N \geq 2p$ and f Schwartz function along $\tilde{Z} = Z^*$, then $\int_{\tilde{Z}=Z^*} f = \int_M DQ\, \mathrm{SDet}^{N/2}(Q)\, F(Q)$ with integration domain $M \cong (\mathrm{GL}_{2p}(\mathbb{R})/\mathrm{O}_{2p}) \times (\mathrm{U}_{2q}/\mathrm{USp}_{2q})$ and $\mathfrak{gl}-$invariant Berezin integration form DQ.

Remark. $(M, \mathcal{A})$ is Riemannian symmetric superspace.

Symmetry argument (heuristic)

Pullback $f = \psi^* F$. Compare two distributions:

Distribution 1: $\mu_1[F] = \int_{\tilde{Z}=Z^*} \psi^* F$

Distribution 2: $\mu_2[F] = \int_M DQ\ \mathrm{SDet}^{N/2}(Q)\, F(Q)$

For $g \in \mathrm{OSp}_{2p|2q}$ let $F_g(Q) := F(gQg^T)$.

The transformation behavior is the same:

$\mu_A[F_g] = \mathrm{SDet}^{-N}(g)\, \mu_A[F] \qquad (A = 1, 2).$

Comments

- Superbosonization field Q is dual to Hubbard-Stratonovich field Q':
  $[Q'] = \text{energy}$, $[Q] = 1/\text{energy}$
- Integration domain for Q independent of energy parameters
- No assumption of Gaussian disorder needed
- Symmetries manifest at all stages of the calculation

Application: Wegner's N-orbital model, U(N)

Hilbert space $V = \oplus_{i\in\Lambda} V_i$, orthogonal projector $\Pi_i : V \to V_i$

Characteristic function: $\int \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\,HK} d\mu(H) = \mathrm{e}^{-(1/2N)\sum_{i,j} C_{ij}\,\mathrm{Tr}\,K\Pi_i K\Pi_j}$

Average ratio of characteristic polynomials:

$$R(E_0, E_1) = \int \frac{\mathrm{Det}\,(E_1 - H)}{\mathrm{Det}\,(E_0 - H)}\, d\mu(H) \qquad (\mathrm{Im}\,E_0 > 0).$$

Hubbard – Stratonovich transformation ($x_k \in \mathbb{R}$, $y_k \in \mathrm{i}\mathbb{R}$):

$$R(E_0, E_1) = \int \mathrm{e}^{-(N/2)\sum_{i,j}(C^{-1})_{ij}(x_i x_j - y_i y_j)} D_{\mathrm{HS}}(x, y) \prod_{k\in\Lambda} \frac{y_k - E_1}{x_k - E_0} \frac{dx_k dy_k}{2\pi\mathrm{i}}$$

Superbosonization ($x_k \in \mathbb{R}_+$, $y_k \in \mathrm{U}_1$):

$$R(E_0, E_1) = \int \mathrm{e}^{-(N/2)\sum_{i,j} C_{ij}(x_i x_j - y_i y_j)} D_{\mathrm{SB}}(x, y) \prod_{k\in\Lambda} \left(\frac{x_k \mathrm{e}^{\mathrm{i} E_0 x_k}}{y_k \mathrm{e}^{\mathrm{i} E_1 y_k}} \right)^N \frac{dx_k dy_k}{2\pi\mathrm{i}\, x_k y_k}$$