Applied Mathematics

מכון טכנולוגי חולון
Holon Institute of Technology

[ Integrable Theory of Quantum Transport ...
Phys. Rev. Lett. 101, 176804 (2008) ] $\pi^{30}$

# Quantum Transport in Chaotic Cavities and Painlevé Transcendents


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IV Brunel Workshop on Random Matrix Theory, December 20, 2008

- Brief Intro: Painlevé property and appearance of Painlevé transcendents in physics
  - 2D Ising model
  - 1D impenetrable Bose gas
  - Growth models
- New! Painlevé transcendents in quantum transport problems: Cumulants of Landauer conductance
- Landauer conductance and its cumulants: Known results
- Integrable theory of quantum transport in chaotic cavities
- Conclusions / Open problems

## Painlevé transcendents and their appearance in physics

$$y'' = F(x, y, y')$$

Rational function of its arguments — **E. Picard** (1889)

All **movable singularities** are restricted to **poles**
**(no movable branch points)**

50 → 44

(a) **linear 2nd order DEs**
(b) **Weierstrass DE**
(c) **Riccati DE**

$$(y')^2 = 4y^3 - g_2 y - g_3$$

$$y' = a(x)\, y^2 + b(x)\, y + c(x)$$

[ 6 — **Painlevé equations $P_I$ – $P_{VI}$**
nonlinear special functions ]

**P. Painlevé** (1900,1902)
**B. Gambier** (1905)
**R. Fuchs** (1910)

## Painlevé transcendents and their appearance in physics

**σP$_{\rm III}$**

$$(t\sigma''_{\rm III})^2 - \nu_1\nu_2\,(\sigma'_{\rm III})^2 + \sigma'_{\rm III}(4\sigma'_{\rm III}-1)(\sigma_{\rm III}-t\sigma'_{\rm III}) - \frac{1}{4^3}\,(\nu_1-\nu_2)^2 = 0$$

**σP$_{\rm V}$**

$$(t\sigma''_{\rm V})^2 - \left[\sigma_{\rm V} - t\sigma'_{\rm V} + 2(\sigma'_{\rm V})^2 + (\nu_0+\nu_1+\nu_2+\nu_3)\sigma'_{\rm V}\right]^2 + 4(\nu_0+\sigma'_{\rm V})(\nu_1+\sigma'_{\rm V})(\nu_2+\sigma'_{\rm V})(\nu_3+\sigma'_{\rm V}) = 0$$

fascinating properties

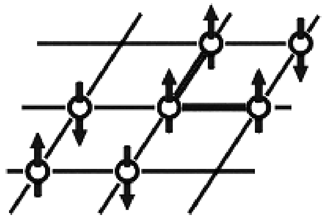
[ 6 Painlevé equations P$_{\rm I}$ – P$_{\rm VI}$ nonlinear special functions ]

P. Painlevé (1900,1902)
B. Gambier (1905)
R. Fuchs (1910)

P. Clarkson, Painleve equations – nonlinear special functions, J. Comp. Appl. Math. 153, 127 (2003)

▸ Painlevé transcendents and their appearance in physics

- 2D Ising model



$$H^{(2D)}_{\text{int}} = -J \sum_{j,k} \left( \boldsymbol{\sigma}_{j,k}\, \boldsymbol{\sigma}_{j,k+1} + \boldsymbol{\sigma}_{j,k}\, \boldsymbol{\sigma}_{j+1,k} \right)$$

T. Wu, B. McCoy, C. Tracy, and E. Barouch (1976)

$$\langle \boldsymbol{\sigma}_{00} \boldsymbol{\sigma}_{MN} \rangle \Big|_{\substack{T \to T_c^{\pm} \\ R=(M^2+N^2)^{1/2} \to \infty}} = \boldsymbol{F}^{\pm} \left( [\sigma_{\text{III}}];\, r = \frac{R}{\xi(T)} \right)$$

$$\xi(T) = \frac{(T_c/4J)}{|1 - T/T_c|}$$

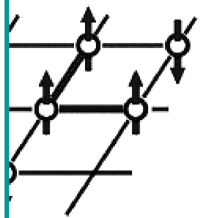
$\sigma P_{III}$

$$\tanh(J/T_c) = \sqrt{2} - 1$$

- Painlevé … cs

  - 2D Isin…

$H_{\text{int}}^{(2\text{D})} = \ldots$



T. Wu, B…

# 2007 Wiener Prize

The committee also recognizes the earlier work of Craig Tracy with Wu, McCoy, and Barouch, in which Painlevé functions appeared for the first time in exactly solvable statistical mechanical models. In addition, the committee recognizes the seminal contributions of Harold Widom to the asymptotic analysis of Toeplitz determinants and their various operator theoretic generalizations.

$$\langle \boldsymbol{\sigma}_{00}\boldsymbol{\sigma}_{MN}\rangle\Big|_{\substack{T\to T_c^{\pm}\\ R=(M^2+N^2)^{1/2}\to\infty}} = \boldsymbol{F}^{\pm}\left([\sigma_{\rm III}];\, r=\frac{R}{\xi(T)}\right)$$

$$\xi(T) = \frac{(T_c/4J)}{|1-T/T_c|}$$

$\sigma P_{III}$

$$\tanh(J/T_c) = \sqrt{2}-1$$

## Painlevé transcendents and their appearance in physics

- **Impenetrable Bose gas** $g \to \infty$

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + g \sum_{i<j} \delta(z_i - z_j)$$

M. Girardeau (1960)
A. Lenard (1964)

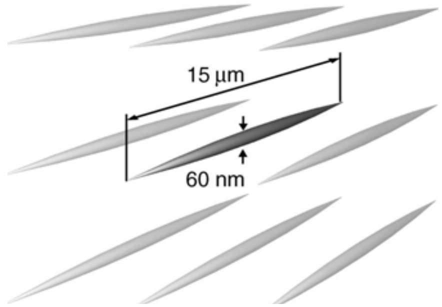



FIG. 1. The geometry and size of trapped 1D gases in a two-dimensional optical lattice. The spacing between the 1D tubes in the horizontal and vertical direction is 413 nm.

© T. Esslinger ETH Zürich

M. Jimbo, T. Miwa, Y. Môri, and M. Sato (1980)

$$\varrho_N(x) = N \int_0^L dz_2 \cdots dz_N\, \Psi^*(x, z_2, \cdots, z_N)\, \Psi(0, z_2, \cdots, z_N)$$

$$\varrho_\infty(x) = \lim_{N\to\infty} \varrho_N(x)\Big|_{L=N} = \exp\left(\int_0^{\pi x} \frac{dt}{t}\, \sigma_{\rm V}(t)\right)$$

$\sigma P_V$

## Painlevé transcendents and their appearance in physics

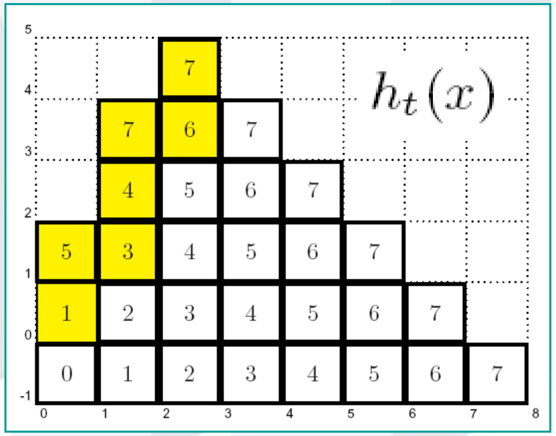
- **Growth models in (1+1)D (oriented digital boiling)**

$$h_{t+1}(x) = \max\left\{h_t(x-1),\, h_t(x) + \varepsilon_{x,t}\right\}$$

$$h_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\varepsilon_{x,t} \sim \mathrm{Ber}(p)$$

J. Gravner, C. Tracy, and H. Widom (2001)




**Universal regime of shape fluctuations**

$$x \to \infty,\ t \to \infty,\ p < p_c = 1 - x/t < 1$$

$$\mathrm{Prob}\left(\frac{h_t(x) - c_1 t}{c_2 t^{1/3}} < s\right) = \boldsymbol{F}_2\left([\sigma_{\mathrm{II}}];\ s\right)$$

$\sigma P_{II}$

## Painlevé transce … ics

- **Growth mode … )**

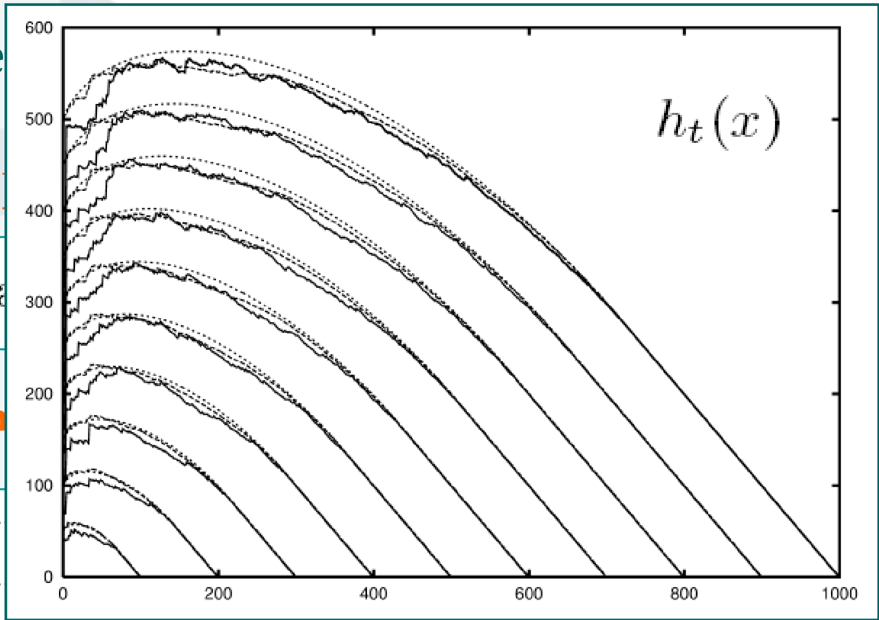
$$h_{t+1}(x) = \mathrm{ma} \ldots$$

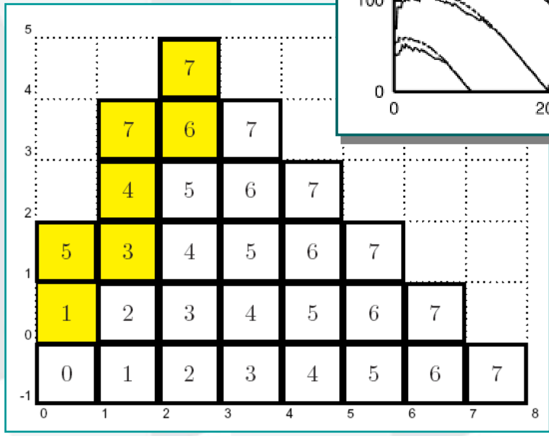
$0$ if $x = 0$

$-\infty$ otherwise

$\sim \mathrm{Ber}(p)$

J. Gravner, C. Tra…



…fluctuations

$$\ldots \to \infty,\ \ldots \to \infty,\ p < p_c = 1 \quad x/t < 1$$

$$\mathrm{Prob}\left(\frac{h_t(x) - c_1 t}{c_2 t^{1/3}} < s\right) = \boldsymbol{F}_2\left([\sigma_{\mathrm{II}}]\,;\ s\right)$$

$\sigma P_{\mathrm{II}}$

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- New! Painlevé transcendents in quantum transport problems: Cumulants of Landauer conductance

# Integrable Theory of Quantum Transport ...

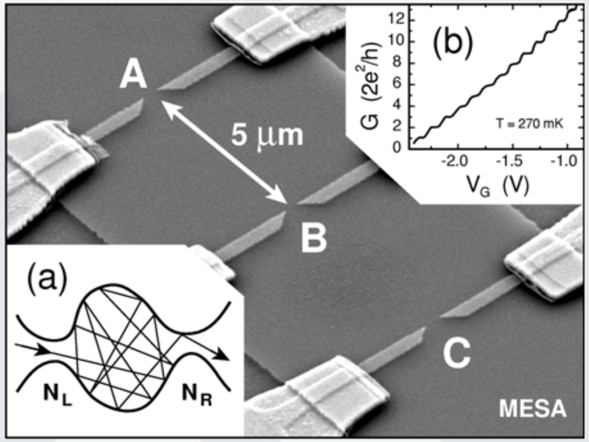
- New: Painlevé in quantum transport !!



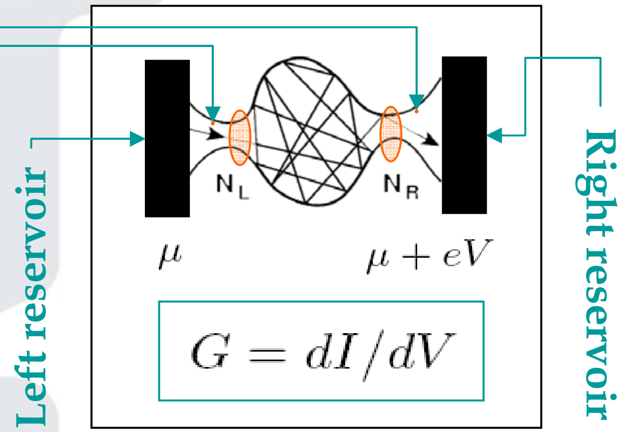
- Painlevé transcendents in quantum transport problems

$$H_{\text{tot}} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell} \psi_\ell + \sum_{\alpha=1}^{N_L+N_R} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M} \sum_{\alpha=1}^{N_L+N_R} \left( \psi_k^\dagger \mathcal{W}_{k\alpha} \chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k \right)$$

cavity — leads — coupling




© S. Oberholzer, Universität Bazel




**Landauer conductance**

# [ Integrable Theory of Quantum Transport … ]

- New: Painlevé in quantum transport !!



- Painlevé transcendents in quantum transport problems

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell} \psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M} \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left( \psi_k^\dagger \mathcal{W}_{k\alpha} \chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k \right)$$

cavity — leads — coupling

**Bohigas Conjecture**

$\{\mathcal{H}_{k\ell}\} \in {\rm GUE}_{M\times M}$

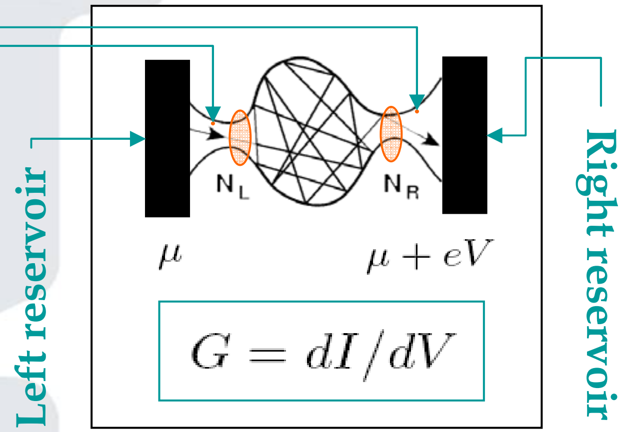
+

**Quantum regime**

$\tau_D \gg \tau_E$

Ehrenfest time

Electron dwell time




**Landauer conductance**

## Painlevé transcendents in quantum transport problems

$$(t\sigma_{\rm V}'')^2 - \left[\sigma_{\rm V} - t\sigma_{\rm V}' + 2(\sigma_{\rm V}')^2 + (N_{\rm L}+N_{\rm R})\sigma_{\rm V}'\right]^2 + 4(\sigma_{\rm V}')^2(\sigma_{\rm V}'+N_{\rm L})(\sigma_{\rm V}'+N_{\rm R}) = 0$$

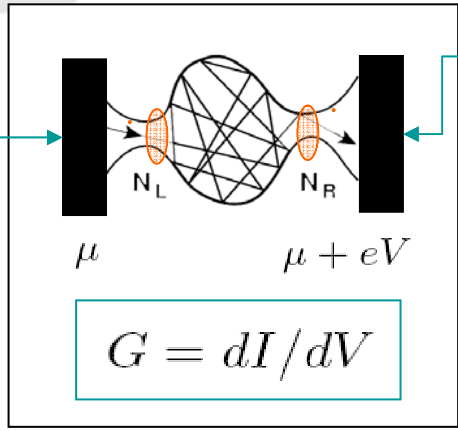
# Main Result

$$\sigma_{\rm V}(z) = N_{\rm L}N_{\rm R} + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\!\langle (G/G_0)^\ell \rangle\!\rangle$$

σ$P_{\rm V}$

$$\kappa_\ell(g) = \langle\!\langle (G/G_0)^\ell \rangle\!\rangle$$

Left reservoir




Right reservoir

## Landauer conductance

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- New! Painlevé transcendents in quantum transport problems: Cumulants of Landauer conductance

- **Landauer conductance and its cumulants: Known results**

## ▸ Landauer conductance and its cumulants: Known results

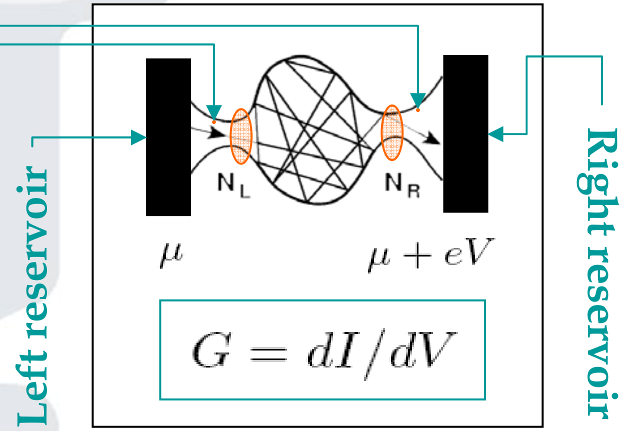
$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k\right)$$

cavity — leads — coupling

### Scattering matrix approach

$$G/G_0 = \mathrm{tr}\,\left(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger\right)$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_{\rm L}} & \\ & 0_{N_{\rm R}} \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} 0_{N_{\rm L}} & \\ & \mathbb{1}_{N_{\rm R}} \end{pmatrix}$$



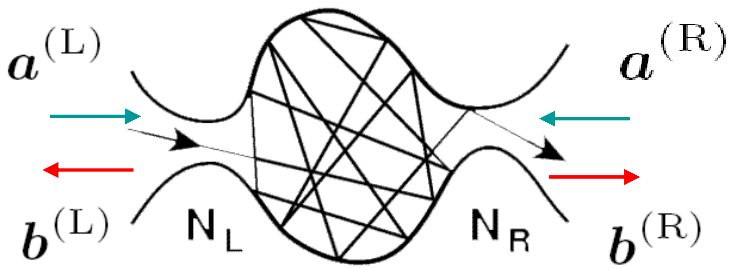

**Landauer conductance**

## Landauer conductance and its cumulants: Known results

$$\mathcal{S}\mathcal{S}^\dagger = \mathbb{1}$$

$$\boldsymbol{a}^{(\mathrm{L})} = \begin{pmatrix} a_1^{(\mathrm{L})} \\ \vdots \\ a_{N_\mathrm{L}}^{(\mathrm{L})} \end{pmatrix} \qquad \boldsymbol{a}^{(\mathrm{R})} = \begin{pmatrix} a_1^{(\mathrm{R})} \\ \vdots \\ a_{N_\mathrm{R}}^{(\mathrm{R})} \end{pmatrix}$$

$$\boldsymbol{b}^{(\mathrm{L})} = \begin{pmatrix} b_1^{(\mathrm{L})} \\ \vdots \\ b_{N_\mathrm{L}}^{(\mathrm{L})} \end{pmatrix} \qquad \boldsymbol{b}^{(\mathrm{R})} = \begin{pmatrix} b_1^{(\mathrm{R})} \\ \vdots \\ b_{N_\mathrm{R}}^{(\mathrm{R})} \end{pmatrix}$$




$$\begin{pmatrix} \boldsymbol{b}^{(\mathrm{L})} \\ \boldsymbol{b}^{(\mathrm{R})} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \boldsymbol{a}^{(\mathrm{L})} \\ \boldsymbol{a}^{(\mathrm{R})} \end{pmatrix}$$

$U_{N_\mathrm{R}}$ $\mathbb{1}_{N_\mathrm{R}}$

**Landauer conductance**

- Landau… …lts

$$a^{(L)} = \qquad b^{(L)} =$$

$$\mathcal{S}(\varepsilon_F) = \mathbb{1} - 2i\pi \mathcal{W}^\dagger (\varepsilon_F - \mathcal{H} + i\pi \mathcal{W}\mathcal{W}^\dagger)^{-1} \mathcal{W}$$

$$\mathcal{H} \in \mathrm{GUE}_{M\times M}$$

$$P(\mathcal{S}) \propto \left[\det\left(\mathbb{1} - \bar{\mathcal{S}}\mathcal{S}^\dagger\right)\det\left(\mathbb{1} - \mathcal{S}\bar{\mathcal{S}}^\dagger\right)\right]^{-(N_\mathrm{L}+N_\mathrm{R})}$$

$$\begin{pmatrix} a_1^{(R)} \\ \vdots \\ a_{N_R}^{(R)} \end{pmatrix} \qquad \begin{pmatrix} b_1^{(R)} \\ \vdots \\ b_{N_R}^{(R)} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{b}^{(L)} \\ \boldsymbol{b}^{(R)} \end{pmatrix} = \mathcal{S} \begin{pmatrix} \boldsymbol{a}^{(L)} \\ \boldsymbol{a}^{(R)} \end{pmatrix}$$

**Landauer conductance**

## Landauer conductance and its cumulants: Known results

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k\right)$$

$$\mathcal{S} \in {\rm CUE}(N_{\rm L} + N_{\rm R})$$

BPC

Scattering matrix approach

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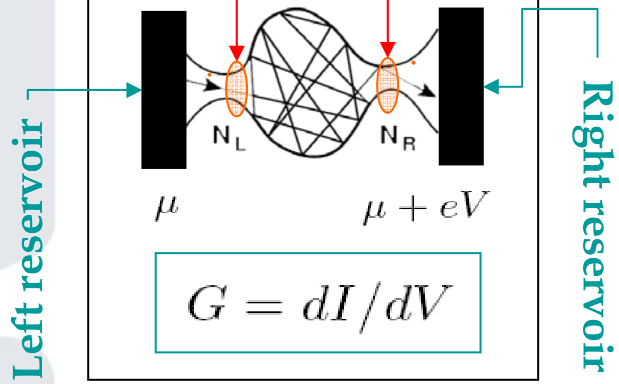


$$G = dI/dV$$

Landauer conductance

## ▸ Landauer conductance and its cumulants: Known results

$$H_{\mathrm{tot}} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell} \psi_\ell + \sum_{\alpha=1}^{N_{\mathrm{L}}+N_{\mathrm{R}}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M} \sum_{\alpha=1}^{N_{\mathrm{L}}+N_{\mathrm{R}}} \left( \psi_k^\dagger \mathcal{W}_{k\alpha} \chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k \right)$$

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**Scattering matrix approach**

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**Semiclassical arguments:**
Blümel & Smilansky (1990)

**Microscopic justification:**
Brouwer (1995)

**Early (exact) calculation of moments/cumulants:**
Baranger & Mello (1994)
**$1^{\mathrm{st}}$ & $2^{\mathrm{nd}}$ cumulants**

## Landauer conductance and its cumulants: Known results

$$\langle g^n \rangle = n\,! \sum_{\lambda \vdash n} \frac{[N_1]_\lambda [N_2]_\lambda}{[N]_\lambda H_\lambda^2}$$

$$\mathcal{S} \in \mathrm{CUE}(N_\mathrm{L} + N_\mathrm{R})$$

$$\sim \exp\left(\pi\sqrt{2n/3}\right)$$

exponential growth !!

**Symmetric functions**

**All moments**
Novaes (2008)

**Selberg integral**

**3rd & 4th cumulants**
Savin, Sommers & Wieczorek (2007)

**Semiclassical arguments:**
Blümel & Smilansky (1990)

**Microscopic justification:**
Brouwer (1995)

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**1st & 2nd cumulants**

# [ Integrable Theory of Quantum Transport … ]
## Outline



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- Landauer conductance and its cumulants: Known results
- Integrable theory of quantum transport in chaotic cavities

## ‣ Integrable theory of quantum transport (Landauer conductance)

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k\right)$$

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⇐ BPC

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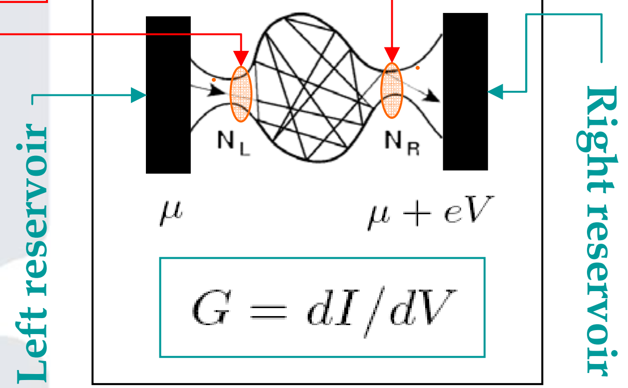



$$G = dI/dV$$

**Landauer conductance**

## ‣ Integrable theory of quantum transport (Landauer conductance)

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in\mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

**Cumulant generating function**

$$n = \min(N_\mathrm{L}, N_\mathrm{R})$$

**Itzykson-Zuber formula, but: high degeneracy of C-matrices**

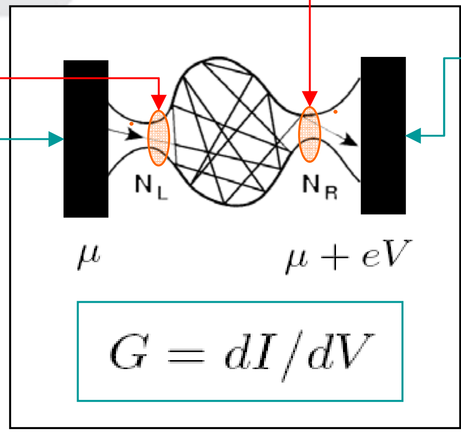
$$\mathcal{S} = \begin{pmatrix} r_{N_\mathrm{L}\times N_\mathrm{L}} & t'_{N_\mathrm{L}\times N_\mathrm{R}} \\ t_{N_\mathrm{R}\times N_\mathrm{L}} & r'_{N_\mathrm{R}\times N_\mathrm{R}} \end{pmatrix}$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_\mathrm{L}} & \\ & 0_{N_\mathrm{R}} \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} 0_{N_\mathrm{L}} & \\ & \mathbb{1}_{N_\mathrm{R}} \end{pmatrix}$$

**Truncate!**

(Zyczkowski & Sommers, 2000)

**BPC**




**Landauer conductance**

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$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in\mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

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$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\boldsymbol{t}\boldsymbol{t}^\dagger)} \right\rangle_{\boldsymbol{t}_{N_\mathrm{R}\times N_\mathrm{L}}}$$

**Truncate!**

(Zyczkowski & Sommers, 2000)

$$\mathcal{F}_n(z) \propto \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^\nu \exp(-zT_j)\cdot \Delta_n^2(\boldsymbol{T})$$

$$\mathcal{S} \in \mathrm{CUE}(N_\mathrm{L} + N_\mathrm{R})$$

‣ **Integrable theory of quantum transport (Landauer conductance)**

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in\mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

**Cumulant generating function**

$n = \min(N_\mathrm{L}, N_\mathrm{R})$

$\nu = |N_\mathrm{L} - N_\mathrm{R}|$

$$\mathcal{F}_n(z) = \exp\left(\int_0^z dt \frac{\sigma_\mathrm{V}(t) - N_\mathrm{L}N_\mathrm{R}}{t}\right)$$

$$\mathcal{F}_n(z) \propto z^{-n(n+\nu)} \int_{(0,z)^n} \prod_{j=1}^n d\lambda_j\, \lambda_j^\nu\, e^{-\lambda_j} \cdot \Delta_n^2(\boldsymbol{\lambda})$$

**Gap formation probability (LUE)**

Tracy & Widom (1994)

$$\mathcal{F}_n(z) \propto \int_{(0,1)^n} \prod_{j=1}^n dT_j\, T_j^\nu \exp(-zT_j) \cdot \Delta_n^2(\boldsymbol{T})$$

$\mathcal{S} \in \mathrm{CUE}(N_\mathrm{L} + N_\mathrm{R})$

## ▸ Integrable theory of quantum transport (Landauer conductance)

**Cumulant generating function**

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in\mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

$$\mathcal{F}_n(z) = \exp\left(\int_0^z dt\,\frac{\sigma_\mathrm{V}(t) - N_\mathrm{L}N_\mathrm{R}}{t}\right)$$

$$\sigma_\mathrm{V}(z) = N_\mathrm{L}N_\mathrm{R} + \sum_{\ell=1}^{\infty}\frac{(-z)^\ell}{\Gamma(\ell)}\langle\langle (G/G_0)^\ell\rangle\rangle$$

!

$$\log\mathcal{F}_n(z) = \sum_{\ell=1}^{\infty}\frac{(-z)^\ell}{\Gamma(\ell+1)}\langle\langle (G/G_0)^\ell\rangle\rangle$$

**Main Result**

$$(t\sigma_\mathrm{V}'')^2 - \left[\sigma_\mathrm{V} - t\sigma_\mathrm{V}' + 2(\sigma_\mathrm{V}')^2 + (N_\mathrm{L}+N_\mathrm{R})\sigma_\mathrm{V}'\right]^2 + 4(\sigma_\mathrm{V}')^2(\sigma_\mathrm{V}'+N_\mathrm{L})(\sigma_\mathrm{V}'+N_\mathrm{R}) = 0$$

- Integrable theory of quantum transport (Landauer conductance)

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k\right)$$

cavity — leads — coupling

**Bohigas Conjecture**

$\{\mathcal{H}_{k\ell}\} \in \mathrm{GUE}_{M\times M}$

+

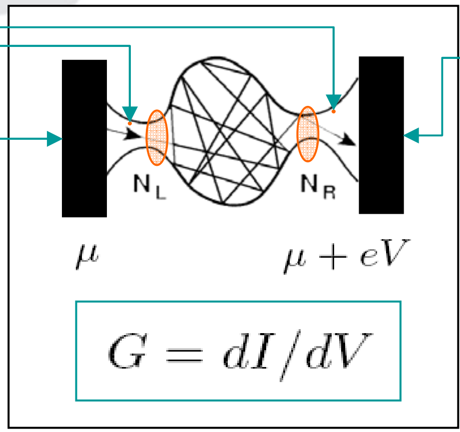
**Quantum regime**

$\tau_D \gg \tau_E$

Ehrenfest time

Electron dwell time

Left reservoir




Right reservoir

**Landauer conductance**

## Painlevé transcendents in quantum transport problems

$$(t\sigma_{\rm V}'')^2 - \left[\sigma_{\rm V} - t\sigma_{\rm V}' + 2(\sigma_{\rm V}')^2 + (N_{\rm L}+N_{\rm R})\sigma_{\rm V}'\right]^2 + 4(\sigma_{\rm V}')^2(\sigma_{\rm V}'+N_{\rm L})(\sigma_{\rm V}'+N_{\rm R}) = 0$$

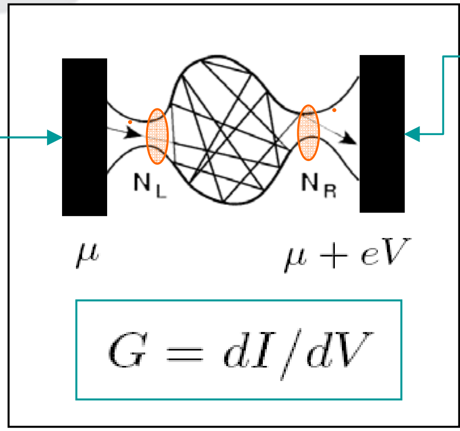
# Main Result

$$\sigma_{\rm V}(z) = N_{\rm L}N_{\rm R} + \sum_{\ell=1}^{\infty} \frac{(-z)^{\ell}}{\Gamma(\ell)} \langle\langle (G/G_0)^{\ell} \rangle\rangle$$

**σP$_{\rm V}$**

$$\kappa_\ell(g) = \langle\langle (G/G_0)^{\ell} \rangle\rangle$$

Left reservoir




Right reservoir

**Landauer conductance**

## Consequences / further results

- **Conductance cumulants obey a nonlinear recurrence equation**

$$[(N_L+N_R)^2 - j^2](j+1)\kappa_{j+1}(g) = 2\sum_{\ell=0}^{j-1}(3\ell+1)(j-\ell)^2\binom{j}{\ell}\kappa_{\ell+1}(g)\kappa_{j-\ell}(g) - (N_L+N_R)(2j-1)\,j\kappa_j(g) - j(j-1)(j-2)\,\kappa_{j-1}(g)$$

$$\kappa_1(g) = \frac{N_L N_R}{N_L+N_R}$$

$$\kappa_2(g) = \frac{\kappa_1^2(g)}{(N_L+N_R)^2-1}$$

$$\Rightarrow \quad \kappa_\ell(g) = P_\ell\left(\kappa_1(g); N_L+N_R\right)$$

$$\langle g^n\rangle = n\,!\sum_{\lambda\vdash n}\frac{[N_1]_\lambda [N_2]_\lambda}{[N]_\lambda H_\lambda^2}$$

$$\sim \exp\left(\pi\sqrt{2n/3}\right)$$

Novaes (2008)

# Integrable Theory of Quantum Transport …

- New: Painlevé in quantum transport !!



## Consequences / further results

- Entire conductance distribution function follows from the Toda Lattice

$$\mathcal{F}_n(z)\,\mathcal{F}_n''(z) - \left(\mathcal{F}_n'(z)\right)^2 = \mathrm{var}_n(g)\,\mathcal{F}_{n-1}(z)\,\mathcal{F}_{n+1}(z)$$

$$\mathcal{F}_0(z) = 0$$

$$\mathcal{F}_1(z) = \frac{(\nu+1)!}{z^{\nu+1}}\left(1 - e^{-z}\sum_{\ell=0}^{\nu}\frac{z^\ell}{\ell!}\right)$$

$$f_n(g) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} dz\,\mathcal{F}_n(z)\,e^{g\,z}$$

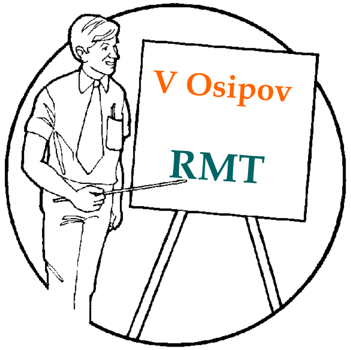
Conductance probability density function

**Statistics of thermal to shot noise crossover in chaotic cavities**

Eugene Kanzieper[1] and Vladimir Al. Osipov[2]
[1] *Department of Applied Mathematics, H.I.T.—Holon Institute of Technology, Holon 58102, Israel*
[2] *Fachbereich Physik, Universität Duisburg-Essen, D-47057 Duisburg, Germany*
(Dated: December 20, 2008)

## Consequences / further results

- Asymptotic analysis of conductance cumulants
- Asymptotic analysis of conductance distribution (deviations from the Gaussian law)
- Statistics of the noise power as a function of bias voltage and the temperature
- Joint statistics of Landauer conductance and the noise power
- ...



- Brief Intro: Painlevé property and appearance of Painlevé transcendents in physics
  - 2D Ising model
  - 1D impenetrable Bose gas
  - Growth models
- New! Painlevé transcendents in quantum transport problems: Cumulants of Landauer conductance
- Landauer conductance and its cumulants: Known results
- Integrable theory of quantum transport in chaotic cavities
- Conclusions / Open problems

## Conclusions / Open problems

$$(t\sigma_{\rm V}'')^2 - \left[\sigma_{\rm V} - t\sigma_{\rm V}' + 2(\sigma_{\rm V}')^2 + (N_{\rm L}+N_{\rm R})\sigma_{\rm V}'\right]^2 + 4(\sigma_{\rm V}')^2(\sigma_{\rm V}'+N_{\rm L})(\sigma_{\rm V}'+N_{\rm R}) = 0$$

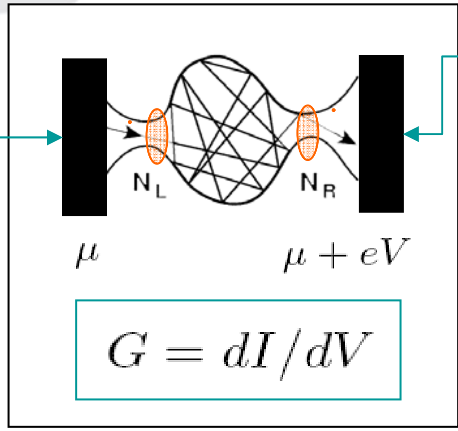
# Main Result

$$\sigma_{\rm V}(z) = N_{\rm L}N_{\rm R} + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\langle (G/G_0)^\ell \rangle\rangle$$

$\sigma P_V$

$$\kappa_\ell(g) = \langle\langle (G/G_0)^\ell \rangle\rangle$$

Left reservoir




Right reservoir

**Landauer conductance**

## Conclusions / Open problems

- Non-ideal contacts (Poisson kernel)

$$P(\mathcal{S}) \propto \left[\det\left(\mathbb{1} - \bar{\mathcal{S}}\mathcal{S}^{\dagger}\right)\det\left(\mathbb{1} - \mathcal{S}\bar{\mathcal{S}}^{\dagger}\right)\right]^{-(N_{\rm L}+N_{\rm R})}$$

- Lossy quantum transport (electrons escaping through the third lead)
- Full counting statistics
- Other symmetry classes ($\beta$=1 and $\beta$=4)