Spectra of Sparse Random Matrices
and Localization on Random Graphs

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Overview

- Look at spectra of sparse symmetric random matrices
  - Follow replica formulation of Edwards and Jones (76), Rodgers and Bray (88)
  - Use techniques recently developed for StatMech of finitely coordinated random systems
  - Use different representation of replica symmetric ansatz
  - Identify DOS of localized and extended states
  - Deconvolution: local DOS of vertices with different coordination

- Explore for various ensembles

- Some details in
  - RK, J Phys A41, 295002, (2008), cond-mat/0803.2886
Spectral Density and Resolvent

- Spectral density of random matrix $M$ from resolvent

$$\rho(\lambda) = \lim_{N \to \infty} \frac{1}{\pi N} \text{Im} \; \text{Tr} \; \left[ \lambda_\varepsilon I - M \right]^{-1}, \quad \lambda_\varepsilon = \lambda - i\varepsilon$$

- express (S F Edwards & R C Jones, JPA, 1976) as

$$\rho(\lambda) = \lim_{N \to \infty} \frac{1}{\pi N} \text{Im} \; \frac{\partial}{\partial \lambda} \text{Tr} \; \ln \left[ \lambda_\varepsilon I - M \right]$$

$$= \lim_{N \to \infty} -\frac{2}{\pi N} \text{Im} \; \frac{\partial}{\partial \lambda} \ln Z_N,$$

where $Z_N$ is a Gaussian integral:

$$Z_N = \int \prod_{i=1}^{N} \frac{du_i}{\sqrt{2\pi / i}} \; \exp \left\{ -\frac{i}{2} \sum_{i,j} u_i (\lambda_\varepsilon \delta_{ij} - M_{ij}) u_j \right\}$$
Sparse Random Matrices

- Sparse symmetric matrix $M$ given, e.g. by

$$M_{ij} = c_{ij}K_{ij}$$

with $\{c_{ij}\}$ adjacency matrix of a random graph. E.g.

$$c_{ij} = \begin{cases} 
0 & \text{with prob } 1 - \frac{c}{N} \\
1 & \text{with prob } \frac{c}{N}
\end{cases}$$

≡ Posisssonian (Erdös Renyi) random graph. Others: regular, scale-free, small-world . . .

- Distribution of $K_{ij}$ arbitrary
  (Gaussian, bimodal, non-random . . .)

Performing the Average — Replica Method

- Replica identity

\[
\ln Z_N = \lim_{n \to 0} \frac{1}{n} \ln Z_N^n
\]

- For integer \( n \), \( Z_N^n \) is partition function of \( n \) identical copies of the system (\( n \)-th power of Gaussian integral)

\[
Z_N^n = \int \prod_{ia} \frac{du_{ia}}{\sqrt{2\pi/i}} \exp \left\{ -\frac{i}{2} \lambda \varepsilon \sum_{i,a} u_{ia}^2 \right. \\
\left. + \frac{c}{2N} \sum_{ij} \left( \exp \left( iK \sum_a u_{ia} u_{ja} \right) \right)_K - 1 \right\}
\]

- Decoupling of sites by introducing the replicated density

\[
\rho(u) = \frac{1}{N} \sum_i \prod_a \delta(u_a - u_{ia})
\]
• Enforce definition via (functional) $\delta$-distribution

$$1 = \int \mathcal{D}\rho \mathcal{D}\hat{\rho} \exp \left\{ -i \int du \hat{\rho}(u) \left( N \rho(u) - \sum_i \prod_a \delta(u_a - u_{ia}) \right) \right\}$$

• Gives

$$\overline{Z_N^n} = \int \mathcal{D}\rho \int \mathcal{D}\hat{\rho} \exp \left\{ N \left[ \frac{c}{2} \int d\rho(u) d\rho(v) \left( \left\langle \exp \left( iK \sum_a u_a v_a \right) \right\rangle_K - 1 \right) \right. \right.$$

$$\left. - \int du i\hat{\rho}(u)\rho(u) + \ln \int \prod_a \frac{du_a}{\sqrt{2\pi/i}} \exp \left( i\hat{\rho}(u) - \frac{i}{2} \lambda \varepsilon \sum_a u_a^2 \right) \right\}$$

• Evaluation of $N^{-1} \ln \overline{Z_N^n}$ by saddle point method
• Stationarity w.r.t. $\rho$ and $\hat{\rho}$

\[
\frac{\delta}{\delta \rho(u)} : \quad i\hat{\rho}(u) = c \int d\rho(v) \left( \langle \exp \left( iK \sum_a u_a v_a \right) \rangle_K - 1 \right) \quad (*)
\]

\[
\frac{\delta}{\delta \hat{\rho}(u)} : \quad \rho(u) = \frac{\exp \left( i\hat{\rho}(u) - \frac{i}{2} \lambda \varepsilon \sum_a u_a^2 \right)}{\int du \exp \left( i\hat{\rho}(u) - \frac{i}{2} \lambda \varepsilon \sum_a u_a^2 \right)} \quad (**)
\]

• Problem: $n \to 0$ limit. (GJ Rodgers, AJ Bray, PRB 37, 1988)

Ansatz: permutation & rotational symmetry in replica space

\[
i\hat{\rho}(u) = cg(|u|)
\]

exploit to perform ‘angular integrals in (*) ,(**)
For $K \in \{\pm 1\}$ get

$$g(u) = -u \int_0^\infty dv \exp \left[ cg(v) - \frac{i}{2} \lambda v^2 \right] J_1(uv), \text{ as } n \to 0$$

Independent SuSy derivation (YV Fyodorov, AD Mirlin, JPA 24, 1991)

Rodgers-Bray Equation extremely difficult to analyze.

Here: different representation of permutation & rotational symmetry. Superpositions of Gaussians:

$$\rho(u) = \int d\pi(\omega) \prod_a \frac{\exp \left[ -\frac{\omega u_a^2}{2} \right]}{Z(\omega)}$$

$$i\hat{\rho}(u) = c \int d\hat{\pi}(\hat{\omega}) \prod_a \frac{\exp \left[ -\frac{\hat{\omega} u_a^2}{2} \right]}{Z(\hat{\omega})}$$

$\Leftrightarrow$ solve (*),(**) in terms of an integral transformation

Get saddle point equations for $\pi$ and $\hat{\pi}$
Population Dynamics

- Self-consistency equations for $\pi$ and $\hat{\pi}$: pair of non-linear integral equations

\[
\hat{\pi}(\omega) = \int d\pi(\omega) \left\langle \delta(\omega - \hat{\Omega}(\omega, K)) \right\rangle_K
\]

\[
\pi(\omega) = \sum_{k \geq 1} \frac{k}{c} p_c(k) \int_{\ell=1}^{k-1} d\hat{\pi}(\hat{\omega}_\ell) \delta(\omega - \Omega_{k-1})
\]

with

\[
\hat{\Omega}(\omega, K) = \frac{K^2}{\omega}, \quad \Omega_{k-1} = i\lambda_\varepsilon + \sum_{\ell=1}^{k-1} \hat{\omega}_\ell
\]

- Structure suggests solving via stochastic population based algorithm; note: get complex $\omega, \hat{\omega}$, but $\text{Re}(\omega) \geq 0, \text{Re}(\hat{\omega}) \geq 0$ selfconsistently in population.
Spectral Density

- Spectral density from solution (using \( \{ \hat{\omega} \}_k = \sum_{\ell=1}^{k} \hat{\omega}_\ell \))

\[
\overline{\rho}(\lambda) = \frac{1}{\pi} \sum_{k=0}^{\infty} p_c(k) \int \prod_{\ell=1}^{k} d\hat{\pi}(\hat{\omega}_\ell) \frac{\text{Re}\{\hat{\omega}\}_k + \varepsilon}{(\text{Re}\{\hat{\omega}\}_k + \varepsilon)^2 + (\lambda + \text{Im}\{\hat{\omega}\}_k)^2}
\]

- With

\[
P(a, b) := \sum_{k} p_c(k) \int \prod_{\ell=1}^{k} d\hat{\pi}(\hat{\omega}_\ell) \delta(a - \text{Re} \{\hat{\omega}\}_k) \delta(b - \text{Im} \{\hat{\omega}\}_k)
\]

get

\[
\overline{\rho}(\lambda) = \int \frac{da \ db}{\pi} P(a, b) \frac{a + \varepsilon}{(a + \varepsilon)^2 + (b + \lambda)^2}.
\]

- Note: singular nature of integrand for \( a = 0 \), as \( \varepsilon \to 0 \):

\[
P(a, b) = P_0(b)\delta(a) + \tilde{P}(a, b)
\]

- Identify localized DOS

\[
\overline{\rho}_{\text{loc}}(\lambda) = P_0(-\lambda)
\]

(R Abou-Chacra, PW Anderson, DJ Thouless, JPC 6, 1973)
Results — Poisson Random Graphs

Spectral densities for $\langle K_{ij}^2 \rangle = 1/c$, on Poissonian random graphs with $c = 4$ (left), and $c = 2$ (right) using $\epsilon = 10^{-300}$ (full line); in both panels: numerical diagonalization results for graphs of size $N = 2000$ (dashed).

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More on the Posisson $c = 2$ case

Upper left: zoom into the central region; upper right: results on logarithmic scale; lower: results regularized at $\varepsilon = 10^{-3}$. In all panels: numerical diagonalization results for graphs of size $N = 2000$ (dashed). Localization for $|\lambda| > 2.295$!
Localization — IPRs

\[ IPR(v) = \frac{\sum_{i=1}^{N} v_i^4}{(\sum_{i=1}^{N} v_i^2)^2} \]

- \( IPR = \mathcal{O}(1) \) for localized, \( \mathcal{O}(N^{-1}) \) for de-localized states

Continuous and full densities of state, and average IPRs for Poissonian random graphs with \( c = 2 \) (left) and \( c = 4 \) (right). Average IPRs from numerical diagonalization of matrices with \( N = 250, N = 500, N = 1000 \) and \( N = 2000 \). Scaling of IPRs confirms location of mobility edges seen in DOS.
Scaling of average IPRs with system size for Poisson Random graphs with $c = 2$ (upper) and $c = 2$ (lower). The fraction of sites not in the giant cluster is $x_i \simeq 0.205$ at $c = 2$ and $x_i \simeq 0.02$ at $c = 4$. 
Other Ensembles

- Poisson random graphs with bimodal couplings

- Regular random graphs with Gaussian or bimodal couplings (recover Wigner semi-circle law in the $c \gg 1$ limit)

- Scale free graphs (power law degree distribution)
  - For $p(k) = P_0 k^{-\gamma}$ confirm $\rho(\lambda) \sim \lambda^{1-2\gamma}$ at large $\lambda$.

- In all cases: localization & mobility edges.
Results — Graph-Laplacians

• Spectra of matrices with row-constraints

\[ M_{ij} = c_{ij} K_{ij} - \delta_{ij} \sum_k c_{ik} K_{ik} \; ; \quad K_{ij} = 1/c \Leftrightarrow M = \Delta \]

Spectral density for the Laplacian on a Poissonian random graph with \( c = 2 \) as computed via the present algorithm. Left: \( \varepsilon = 10^{-3} \)-results; right: results from numerical diagonalisation of \( N \times N \) matrices of the same type with \( N = 2000 \).
Continuous Spectrum and 'Low-Energy' Lifshitz Tail

Spectral density for the Laplacian on a Poissonian random graph with $c = 2$. Left: continuous part of the spectrum obtained using $\varepsilon = 10^{-300}$ as a regularizer. Right: zoom into the small $|\lambda|$ region, exhibiting a mobility edge and a localized DOS ($\varepsilon = 10^{-5}$ and $10^{-6}$) compatible with Lifshitz tail behaviour.
Results — Unfolding Spectral Densities

Spectral density for the Laplacian on a Poissonian random graph with $c = 2$ (full upper line), shown together with its unfolding according to contributions of different coordination. Identifiable humps from left to right: $k = 9, k = 8, \ldots, k = 3$. Several notable humps from $k = 2$, together with the $k = 1$ contribution mainly responsible for dip at $\lambda = -1$. The $k = 0$ contribution is mainly responsible for the $\delta$-peak at $\lambda = 0$. 
Results — Random Schrödinger Operators

- Spectral properties of discrete random Schrödinger operator

\[ H = -\Delta + V, \quad V_{ij} = v_i \delta_{ij}, \quad v_i \in [-W, W] \]

RSO on a Poissonian random graph with \( c = 4 \), and \( W = 1 \). Left: Spectral density and IPRs (\( N = 250, 500, 1000, \) and \( 2000 \)). Right: Continuous DOS and its unfolding (\( k = 1, \ldots, 13 \))
RSO on a Poissonian random graph with $c = 4$, and $W = 1$. DOS and its unfolding ($k = 0, \ldots, 5$).
Summary


- Techniques, ansätze etc inspired by previous work on Stat Mech of heterogeneous systems.

- Allows to disentangle pure point and continuous spectrum.

- Allows to compute local DOS unfolded according to coordination.

- Method is versatile (Poissonian and other degree distributions); Laplacians; discrete random Schrödinger operators; Anderson localisation.

- To do: asymmetric matrices (Rogers, Anand); modular& small world systems; eigenvector distributions, spectral correlations . . .