Eigenvalue Correlations for the Real Ginibre Ensemble

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What is the statistics of eigenvalues Λ_i

of a real $N \times N$ matrix J_{ij}

with i.i.d. Gaussian elements ?

$$d\mu(J_{ij}) = \prod_{ij} \left(\frac{dJ_{ij}}{\sqrt{2\pi}} \, e^{-J_{ij}^2/2} \right)$$

Ginibre 1965

$$\det(J - \Lambda_i) = 0$$

examples

- dynamics of neural networks

$$\dot{h}_i = -h_i + \sum_{j=1}^{N} J_{ij} \tanh(g h_j)$$

Sompolinsky et al. 1988

- directed quantum chaos — Efetov 1997
- quantum chromodynamics — Halasz et al. 1997
- financial markets — Kwapien et.al. 2006
- quantum information theory — Bruzda et.al. arXiv

So, Crisanti, Sompolinsky, Stein (1988)

Spectrum of Large Random Asymmetric Matrices

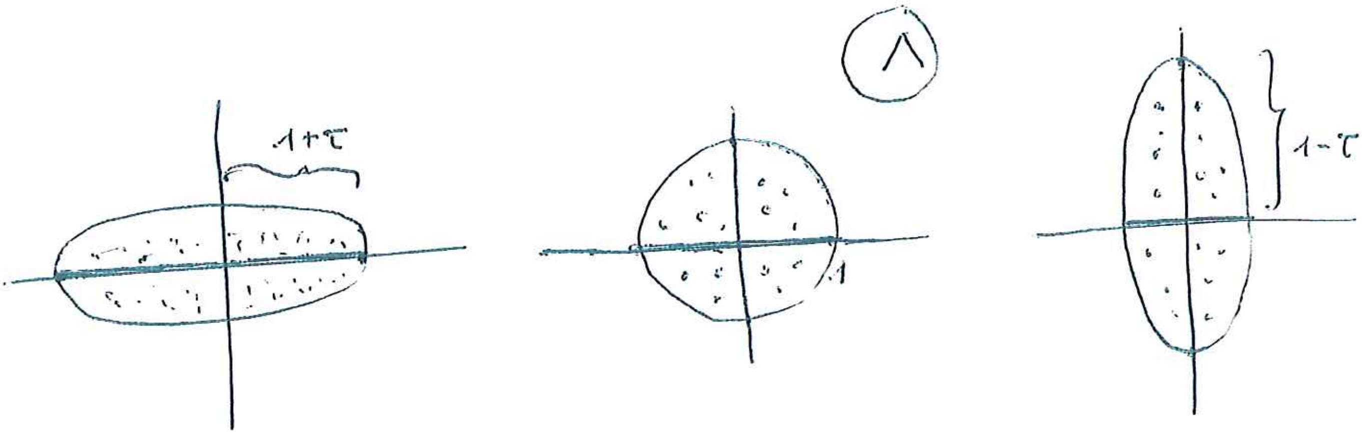
$$\overline{J_{ij}^2} = \frac{1}{N} , \quad \overline{J_{ij} J_{ji}} = \frac{1}{N} \cdot \tau \qquad (i \neq j)$$

with τ: $-1 \leq \tau \leq +1$

$\tau = 0$: total asymmetry

$\tau = 1$: total symmetry (GOE)

$\tau = -1$: total antisymmetry



Λ

$\tau > 0$ $\tau = 0$ $\tau < 0$

excess eigenvalues on real axis $\propto \sqrt{N}$

Hermitian Ensembles : $H = H^+$

$$d\mu(H) \propto DH \; \exp(-\tfrac{1}{2} \operatorname{Tr} H^2)$$

$$\det(H - E_i) = 0$$

j.p.d of eigenvalues E_i:

$$d\mu(E_1, E_2, \dots, E_N) \propto dE_1 \cdots dE_N \prod_{i<j} |E_i - E_j|^\beta \prod_k e^{-E_k^2/2}$$

$$\text{with} \quad \beta = \begin{cases} 1 & \\ 2 & \text{for} \\ 4 & \end{cases} \quad \begin{matrix} \text{GOE} \\ \text{GUE} \\ \text{GSE} \end{matrix}$$

all eigenvalue correlations known:

Pfaffians (determinants)

Gaussian Unitary Ensemble

$$R_n(E_1, E_2, \dots, E_n) = \det\left[K_N(E_j, E_k) \right] \qquad j,k = 1,2,\dots n$$

with

$$K_N(E_1, E_2) = \frac{1}{\sqrt{2\pi}}\, e^{-(E_1^2+E_2^2)/4} \sum_{n=0}^{N-1} P_n(E_1)\cdot P_n(E_2)$$

and

$$\int \frac{dE}{\sqrt{2\pi}}\, e^{-E^2/2}\, P_n(E) P_m(E) = \delta_{n,m}$$

(Hermite - polynomials)

$$R_1(E) = \frac{e^{-E^2/2}}{\sqrt{2\pi}} \sum_{n=0}^{N-1} P_n(E)^2 \qquad = \text{density}$$

$$\simeq \frac{1}{2\pi} \sqrt{4N - E^2} \qquad \text{semicircle law}$$

Complex Ginibre Ensemble

$$d\mu(J) = \prod_{ij} \left(\frac{d^2 J_{ij}}{\pi} e^{-|J_{ij}|^2} \right)$$

$$\det(J - z_j) = 0$$

j.p.d. of eigenvalues z_j :

$$d\mu(z_1, z_2, \dots, z_N) \propto e^{-\sum_i |z_i|^2} \prod_{i<j} |z_i - z_j|^2 \, d^2 z_1 \dots d^2 z_N$$

n-point-densities (correlations)

$$R_n(z_1, \dots, z_N) = \det(K_N(z_j, z_k)) \qquad j, k = 1, 2, \dots n$$

with

$$K_N(z_1, z_2) = \frac{e^{-(|z_1|^2 + |z_2|^2)/2}}{\pi} \sum_{n=0}^{N-1} \frac{(z_1 \bar{z}_2)^n}{n!}$$

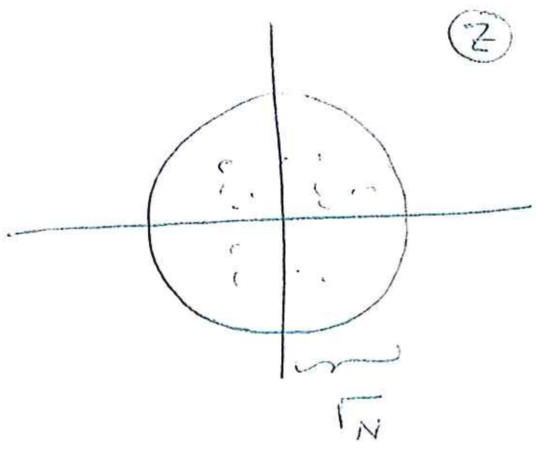
density:

$$R_1(z) = \frac{e^{-|z|^2}}{\pi} \sum_{n=0}^{N-1} \frac{|z|^{2n}}{n!}$$

$$\Rightarrow \frac{1}{\pi} \quad \text{for } N \to \infty \qquad (|z| < \sqrt{N})$$

$$\int_{|z|<\sqrt{N}} R_1(z)\, d^2z = N = \int_{|z|<\sqrt{N}} \frac{d^2z}{\pi}$$

constant in disk of radius $\sqrt{N}$



Correlations

$$R_2(z_1, z_2) = \begin{vmatrix} K_N(z_1, z_1) & K_N(z_1, z_2) \\ K_N(z_2, z_1) & K_N(z_2, z_2) \end{vmatrix}$$

$$\simeq \frac{1}{\pi}\left(1 - e^{-|z_1 - z_2|^2}\right) \text{ for } N \to \infty$$

Cubic level repulsion for $|z_1 - z_2|^2 \to 0$

Cluster function

$$R_1(z_1) R_1(z_2) - R_2(z_1, z_2) \simeq \frac{1}{\pi} e^{-|z_1 - z_2|^2}$$

exponential decay

Results for real Ginibre ensemble ?

Ginibre 1965

Sommers, Crisanti, Sompolinsky, Stein 1988

Lehmann, Sommers 1991

Edelman, Kostlan, Shub 1994

Edelman 1997

Kanzieper, Akemann 2005

Akemann, Kanzieper 2007

Forrester, Nagao 2007

Sommers 2007

Sinclair 2007

Sommers, Wieczorek 2008

Forrester, Nagao 2008

Borodin, Sinclair arXiv

Forrester, Mays arXiv

Akemann, Philips, Sommers arXiv

1. numerical simulation

histogramm of eigenvalues of

real random matrix with $N = 20$

eigenvalues in disk of radius $\sqrt{N} = \sqrt{20}$

δ-function on real axis

2. analytical results

real axis: Edelman, Kostlan and Shub 1994

complex plane: Edelman 1997

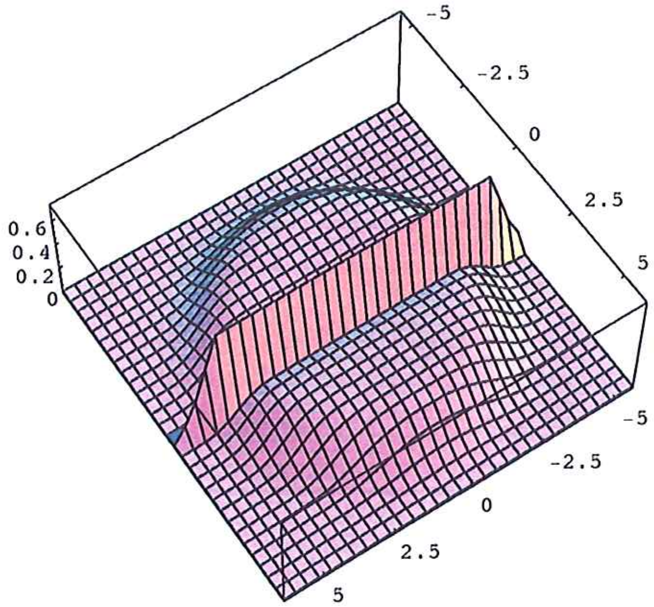
$$R_1^C(\Lambda) = f(\Lambda) \cdot \frac{|\Lambda - \bar{\Lambda}|}{\sqrt{2\pi}} \sum_{n=0}^{N-2} \frac{|\Lambda|^{2n}}{n!}$$

$$\simeq \frac{1}{\pi}\,\theta(\sqrt{N} - |\Lambda|)$$

real axis:

$$R_1^R(\Lambda) \simeq \frac{1}{\sqrt{2\pi}}\,\theta(\sqrt{N} - |x|) \qquad \Lambda = x + iy$$

number of real eigenvalues $N^R(N) \simeq \sqrt{\frac{2N}{\pi}}$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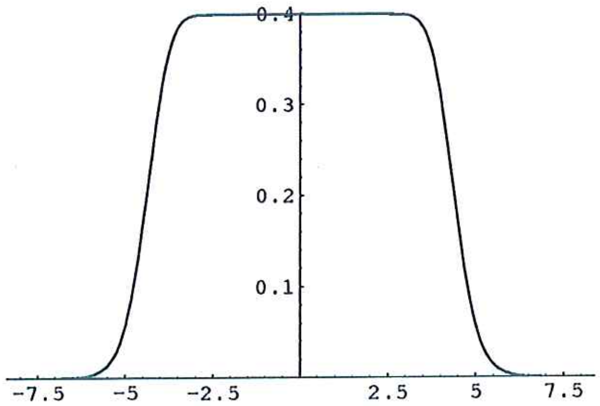

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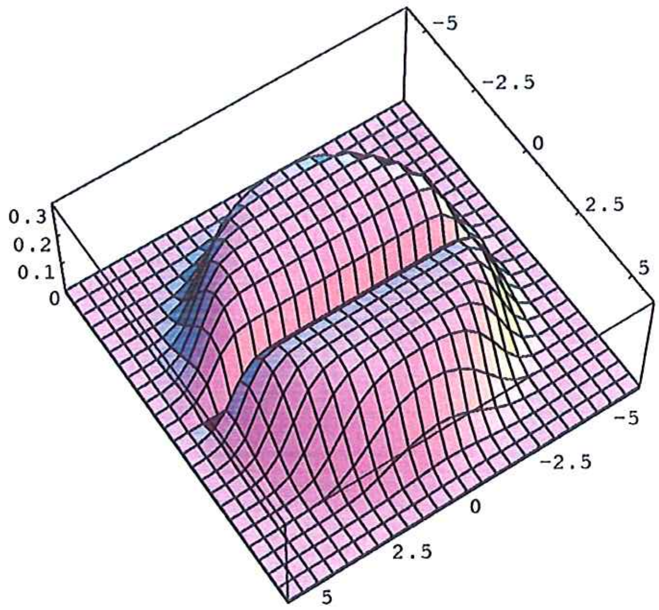
7.5



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-5
-2.5
0
2.5
5

Joint probability density Lehmann, S. 1991

$\det(J - \Lambda_i) = 0$

$$d\mu(\Lambda_1, \Lambda_2, \dots \Lambda_N) = C_N \, d\Lambda_1 \cdots d\Lambda_N \prod_{i<j} (\Lambda_i - \Lambda_j) \prod_k f(\Lambda_k)$$

with

$$f(\Lambda) = e^{-(\Lambda^2 + \bar{\Lambda}^2)/4} \left[\operatorname{erfc}\left(\sqrt{\mathrm{Im}\,\Lambda} \sqrt{2} \right) \right]^{1/2} = f(\bar{\Lambda}) > 0$$

and

$$C_N = VO(N) \cdot 2^{-N} \cdot (2\pi)^{-N(N+1)/4}$$

$$VO(N) = \prod_{d=1}^{N} \frac{2 \cdot \pi^{d/2}}{\Gamma(d/2)} = \text{Volume of } O(N).$$

Λ_i ordered.

$N = 1$: $\Lambda = \bar{\Lambda}$

$$f(\Lambda) = e^{-\Lambda^2/2}$$

$$C_1 = 2 \cdot 2^{-1} \cdot (2\pi)^{-1/2} = \frac{1}{\sqrt{2\pi}}$$

N = 2

a) both eigenvalues real

$$d\mu(\lambda_1, \lambda_2) = C_2 \cdot d\lambda_1 \, d\lambda_2 \; (\lambda_1 - \lambda_2) \; e^{-(\lambda_1^2 + \lambda_2^2)/2} \qquad > 0$$

with $\lambda_1 > \lambda_2$

b) eigenvalues complex conjugate : $\lambda_2 = \bar{\lambda}_1 = \bar{\lambda}$

$$d\mu(\lambda, \bar{\lambda}) = C_2 \cdot d\lambda \cdot d\bar{\lambda} \; (\lambda - \bar{\lambda}) \; e^{-(\lambda^2 + \bar{\lambda}^2)/2} \operatorname{erfc}(|\mathrm{Im}\,\lambda| \sqrt{2})$$

$$= C_2 \; dx \, dy \cdot 4y \cdot e^{-(x^2 - y^2)} \operatorname{erfc}(|y| \sqrt{2})$$

with $y > 0$, $\lambda = x + iy$

$$\int_{\text{real}} d\mu(\lambda_1, \lambda_2) = \frac{1}{\sqrt{2}}$$

$$\int_{\text{complex}} d\mu(\lambda_1, \lambda_2) = 1 - \frac{1}{\sqrt{2}}$$

reason for error function:

$$\begin{bmatrix} a & b+c \\ b-c & d \end{bmatrix}$$ not to diagonal form by orth. trafo

$$\begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}$$ invariant

remaining integration over c ⇒ error function

$\boxed{N > 2}$ bring J with orth. trafo to form

$$\begin{bmatrix} \boxed{\Lambda} & & & 0 \\ & \boxed{\Lambda} & & \\ & & \boxed{\Lambda} & \\ \Delta & & & \ddots \end{bmatrix}$$

decomposes into 2×2 problems

Jacobian $= \prod_{i<j} (\Lambda_i - \Lambda_j)$

for all possible cases, Λ_i ordered

1. all Λ_i real

$\Lambda_1 > \Lambda_2 > \Lambda_3 \cdots$

2. two Λ_i complex conjugate

$\Lambda_1 = \overline{\Lambda_2}$, $\mathrm{Im}\,\Lambda_1 > 0$, $\Lambda_3 > \Lambda_4 > \cdots$

3. four Λ_i complex conjugate

$\Lambda_1 = \overline{\Lambda_2}$, $\Lambda_3 = \overline{\Lambda_4}$, $\mathrm{Im}\,\Lambda_1 > 0$, $\mathrm{Im}\,\Lambda_3 > 0$, $\mathrm{Re}\,\Lambda_1 > \mathrm{Re}\,\Lambda_3$

$\Lambda_5 > \Lambda_6 > \cdots$

and so on.

together with $d\Lambda_1 d\Lambda_2 \cdots d\Lambda_N$: measure positive

n - point densities

a) numerical simulation $N = 20$

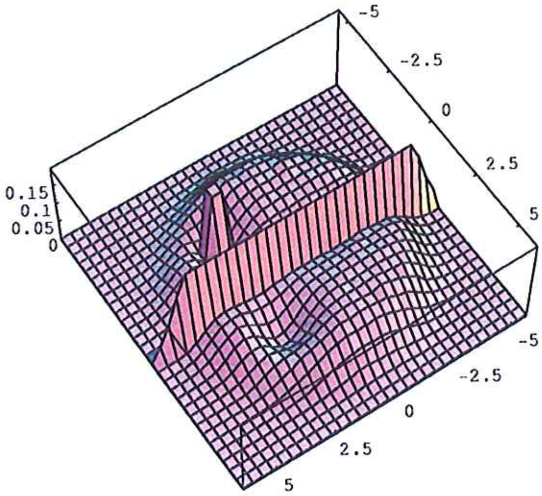
histogramm of pairs (z_1, z_2) with

fixed $z_2 = 2+2i$, $z_2 = 2 + 0{,}5\, i$

eigenvalue z_2 inside hole : cubic level repulsion

δ - function on real axis : mixed correlations

below: 3-point density with z_2, z_3 complex, fixed.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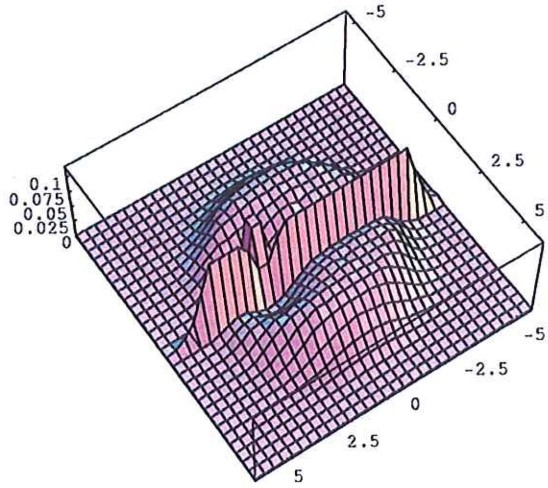
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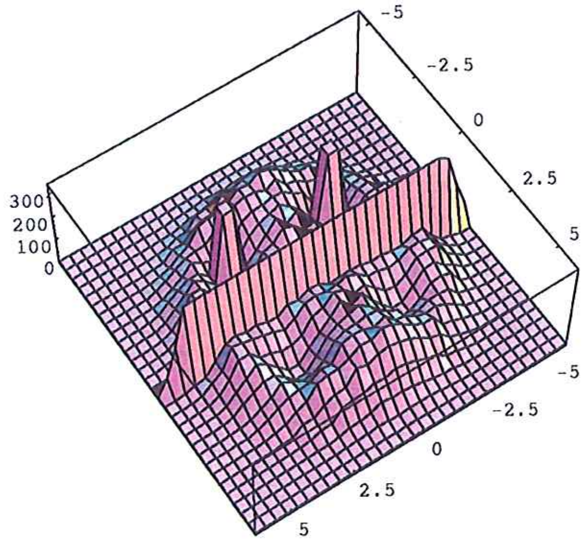
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200
100
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-5
-2.5
0
2.5
5

b) analytical results

above: eigenvalue density on real axis

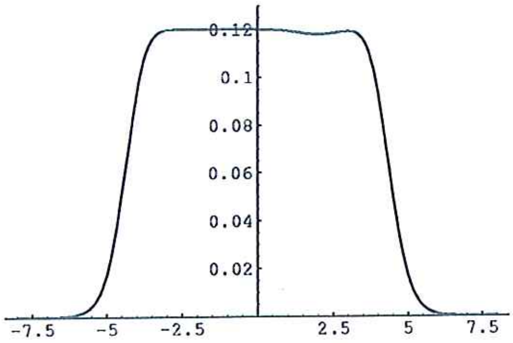
impact of z_L: repulsion

middle: holes agglomerate

below: 2-point correlation on real axis

for $N = 2, 3, 4, 5, 6, \dots$

$$R_2^R(x,0) = \frac{1}{\sqrt{2\pi}} R_1^R(x) - \frac{1}{2\pi} e^{-x^2} + \frac{|x|}{2\sqrt{2\pi}} e^{-\frac{x^2}{2}} \operatorname{erfc}\left(\frac{|x|}{\sqrt{2}}\right)$$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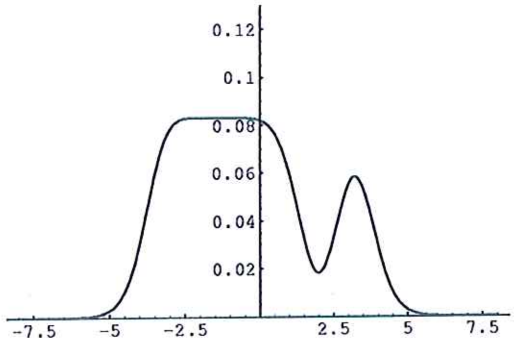
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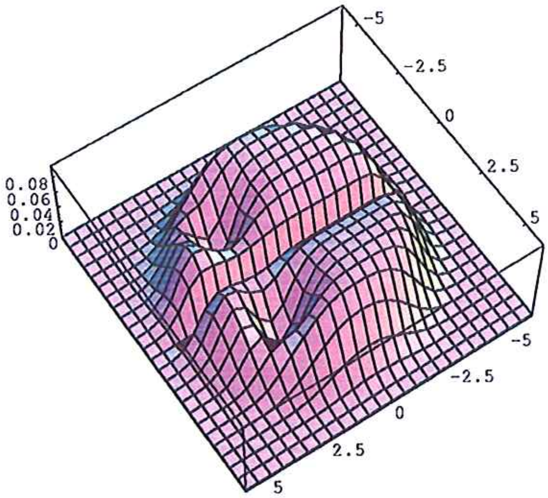
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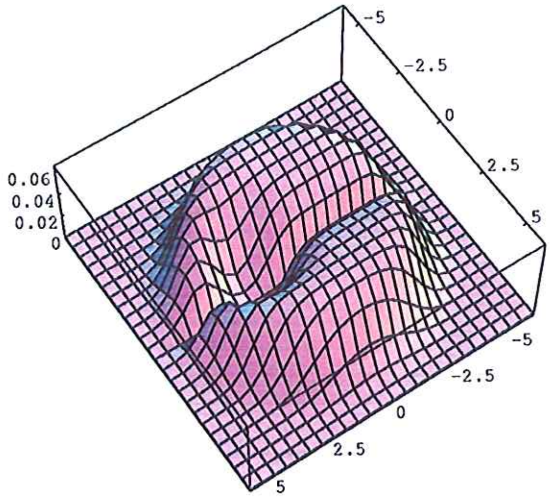
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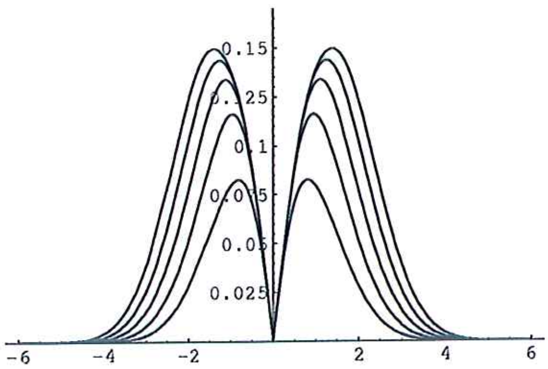
0.15
0.025
-6
-4
-2
2
4
6

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Generating functional for n-point-densities

$z = \lambda = x + iy \quad , \quad d^2 z = dx\, dy$

$$R_n(z_1, \dots, z_n) = \frac{\delta}{\delta g(z_1)} \cdots \frac{\delta}{\delta g(z_n)} Z[g] \Big|_{g(z) \equiv 1}$$

$$Z[g] = \int d^2 z_1 \cdots d^2 z_N \; g(z_1) \cdots g(z_N) \; P(z_1, \dots z_N)$$

$$= C_N \int d^2 z_1 \cdots d^2 z_N \cdot \prod_{i=1}^{N} g(z_i) f(z_i) \cdot \prod_{k<j} (z_k - z_j) \cdot$$

$$\Big\{ \delta(y_1) \cdots \delta(y_N) \; \theta(x_1 > x_2 \cdots > x_N)$$

$$+ (-2i)\, \theta(y_1)\, \delta^2(z_1 - \bar{z}_2)\; \delta(y_3) \cdots \delta(y_N)\; \theta(x_3 > \cdots x_N)$$

$$+ (-2i)^2\, \theta(y_1)\, \theta(y_3)\, \delta^2(z_1 - \bar{z}_2)\; \delta^2(z_3 - \bar{z}_4)\; \theta(x_1 - x_3)$$

$$\cdot\, \delta(y_5)\, \delta(y_6) \cdots \delta(y_N)\; \theta(x_5 > \cdots > x_N)$$

$$+ \cdots \Big\}$$

integrand automatically symmetrized.

Vandermonde determinant:

$$\prod_{(i<j)} (z_i - z_j) = (-1)^{\frac{N(N-1)}{2}} \det (z_1^{k-1}, z_2^{k-1}, \dots, z_N^{k-1})$$

$$= (-1)^{\frac{N(N-1)}{2}} \int d\eta_1^* d\eta_1 \dots d\eta_N^* d\eta_N \exp(-\sum_{k\ell} \eta_k^* z_\ell^{k-1} \eta_\ell)$$

$$= \int d\eta_1^* d\eta_2^* \dots d\eta_N^* \left(\sum_k \eta_k^* z_1^{k-1} \right) (\dots) \left(\sum_k \eta_k^* z_N^{k-1} \right)$$

$\Rightarrow$

$$Z[g] = C_N \int d\eta_1^* d\eta_2^* d\eta_N^* \exp(-\tfrac{1}{2} \sum_{k\ell} \eta_k^* \tilde{A}_{k\ell} \eta_\ell^*)$$

$$= C_N \,\mathrm{Pfaff}\,[\tilde{A}_{k\ell}] \qquad k, \ell = 1, 2, \dots N$$

$$= C_N \sqrt{\det (\tilde{A}_{k\ell})}$$

with

$$\tilde{A}_{k\ell} = \int d^2 z_1 \, d^2 z_2 \, g(z_1) g(z_2) \; \mathcal{F}(z_1, z_2) \, z_1^{k-1} z_2^{\ell-1} = -\tilde{A}_{\ell k}$$

and

$$\mathcal{F}(z_1, z_2) = \left[2i \, \delta^2(z_1 - \bar{z}_2) \,\mathrm{sgn}\, y_1 + \delta(y_1) \delta(y_2) \,\mathrm{sgn}(x_2 - x_1) \right] \rho(z_1) \rho(z_2)$$

$$= - \mathcal{F}(z_2, z_1)$$

additive contributions in Pfaffian

$$A_{kl} = \tilde{A}_{kl} \Big|_{g \equiv 1}$$

$$K_N(z_1, z_2) = \sum_{k,l=1}^{N} A^{-1}_{kl}\, z_1^{k-1} z_2^{l-1} = - K_N(z_2, z_1)$$

1-point density

$$R_1(z) = \frac{\delta Z[g]}{\delta g(z)} \Big|_{g\equiv 1} = \frac{1}{2} \frac{\delta \ln \det \tilde{A}}{\delta g(z)} \Big|_{g \equiv 1}$$

$$= \frac{1}{2} \mathrm{Tr}\, \tilde{A}^{-1} \frac{\delta \tilde{A}}{\delta g(z)} \Big|_{g \equiv 1}$$

$$= \int d^2 z_2\, \mathcal{F}(z, z_2)\, K_N(z_2, z)$$

$$= R_1^C(z) + \delta(y)\, R_1^R(x)$$

complex real

Compare $R_1^c(z)$ with Edelman's result:

$$K_N(z_2, z_1) = \frac{z_2 - z_1}{2\sqrt{2\pi}} \sum_{n=0}^{N-2} \frac{(z_1 z_2)^n}{n!}$$

$$= \sum_{k,\ell}^{1..N} A^{-1}_{k\ell}\, z_2^{k-1} z_1^{\ell-1}$$

$$A^{-1}_{k\ell} = \frac{1}{2\sqrt{2\pi}} \begin{bmatrix} 0 & -\frac{1}{0!} & 0 & \cdots & & 0 \\ \frac{1}{0!} & 0 & -\frac{1}{1!} & \cdots & & \\ 0 & \frac{1}{1!} & 0 & -\frac{1}{2!} & & \\ & 0 & \frac{1}{2!} & 0 & \ddots & \\ & & & \ddots & \ddots & \end{bmatrix}$$

$$R_2(z_1, z_2) = \frac{\delta^2}{\delta g(z_1)\, \delta g(z_2)} Z[g] \Big|_{g \equiv 1} \Longrightarrow$$

$$= F(z_1, z_2) K_N(z_2, z_1) + \int d^2 z_3\, d^2 z_4 \Big\{$$

$$+ F(z_1, z_3) K_N(z_3, z_1)\, F(z_2, z_4)\, K_N(z_4, z_2)$$

$$- F(z_1, z_3) K_N(z_3, z_2)\, F(z_2, z_4)\, K_N(z_4, z_1)$$

$$- F(z_1, z_3) K_N(z_3, z_4)\, F(z_4, z_2)\, K_N(z_2, z_1) \Big\}$$

Diagram expansion w.r.t. $u(z) = g(z) - 1$

(Wick-theorem for fermions)

$\Rightarrow$ all n-point densities can be expressed as Pfaffians:

$$R_n(z_1, z_2, \dots, z_n) = \left| \operatorname{Pfaff} \begin{bmatrix} K & -G \\ G^T & -W \end{bmatrix} \right|$$

with $K_{ij} = K_N(z_i, z_j)$ $\qquad i,j = 1, 2, \dots n$

$$G_{ij} = -\int d^2z \, K_N(z_i, z) \, \mathcal{F}(z, z_j)$$

$$W_{ij} = -\mathcal{F}(z_i, z_j) + \int d^2z \int d^2z' \, \mathcal{F}(z_i, z) \cdot$$
$$\cdot K_N(z, z') \, \mathcal{F}(z', z_j)$$

Results can be written in form, which is analytic in N, and one can prove, that these are then even valid for odd N too.

e.g. with help of incomplete Gamma function:

$$\gamma^*(n,x) = \frac{1}{\Gamma(n)\,x^n} \cdot \int_0^x du\, e^{-u}\, u^{n-1}$$

$$= e^{-x} \sum_{m=0}^{\infty} \frac{x^m}{\Gamma(n+m+1)}$$ is analytic in n and x

$\Rightarrow$

$$K_N(z_1,z_2) = \frac{z_1-z_2}{2\sqrt{2\pi}}\, e^{z_1 z_2} \left(1-(z_1 z_2)^{N-1}\, \gamma^*(N-1, z_1 z_2)\right)$$

$$R_1^R(z_1) = \frac{1}{\sqrt{2\pi}} \left(1 - x_1^{2(N-1)}\, \gamma^*(N-1, x_1^2)\right)$$

$$+ \frac{e^{-x_1^2/2}\, x_1^{2(N-1)}}{2^{N-1/2}\, \Gamma(N/2)} \cdot \gamma^*\left(\frac{N-1}{2}, \frac{x_1^2}{2}\right)$$

Other correlations similar