Superstatistical Generalisations of Wishart-Laguerre ensembles

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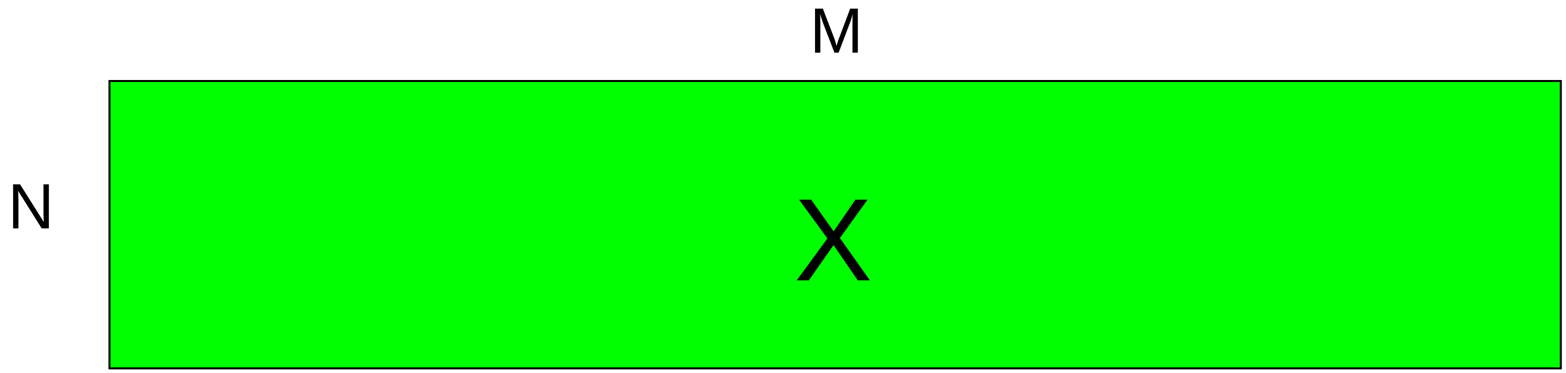
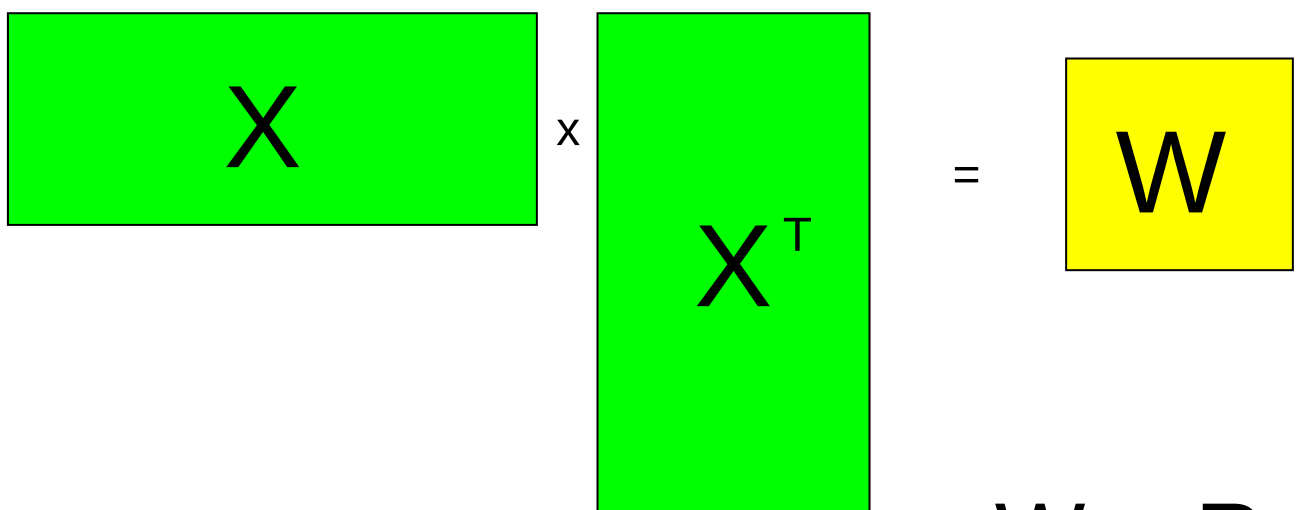


$$\mathbf{X}_{ij} \sim \mathcal{N}(0,1)$$



W = Random Covariance Matrix

Wishart-Laguerre ensemble

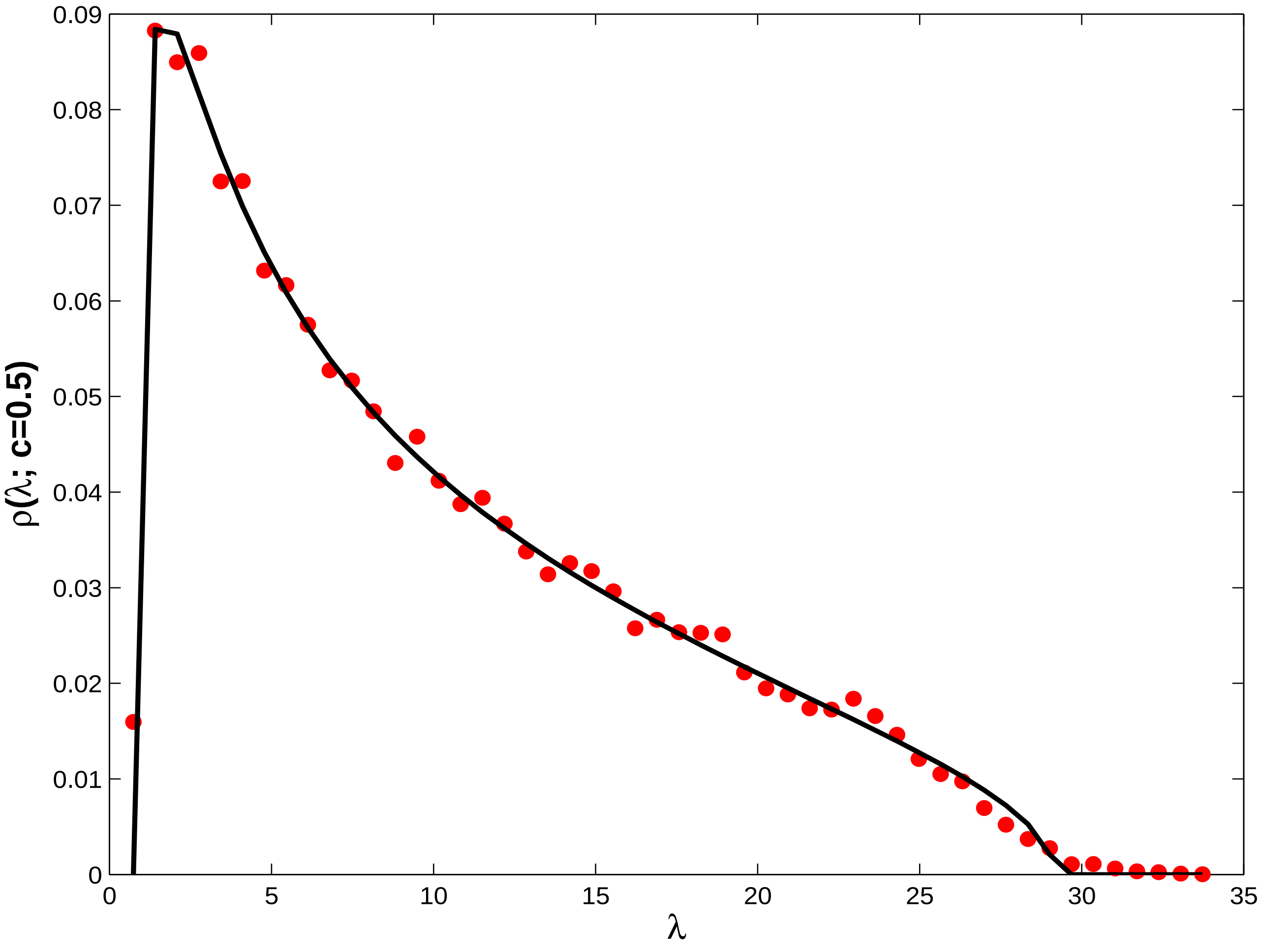
- Introduced by John Wishart in 1928
- JPD of eigenvalues (real and positive) is known

$$P(\lambda_1,\ldots,\lambda_N) = C_N \prod_{i=1}^{N} e^{-\frac{1}{2}\lambda_i} \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^{\beta}$$

- Density of eigenvalues for $N, M \to \infty; N/M = c$

$$\begin{aligned} \rho(\lambda; c) &= \beta N^{-1} f(\beta N^{-1}\lambda) \\ f(x) &= \frac{1}{2\pi x}\sqrt{(x - X_-)(X_+ - x)} \end{aligned}$$

Marcenko-Pastur distribution



Superstatistics

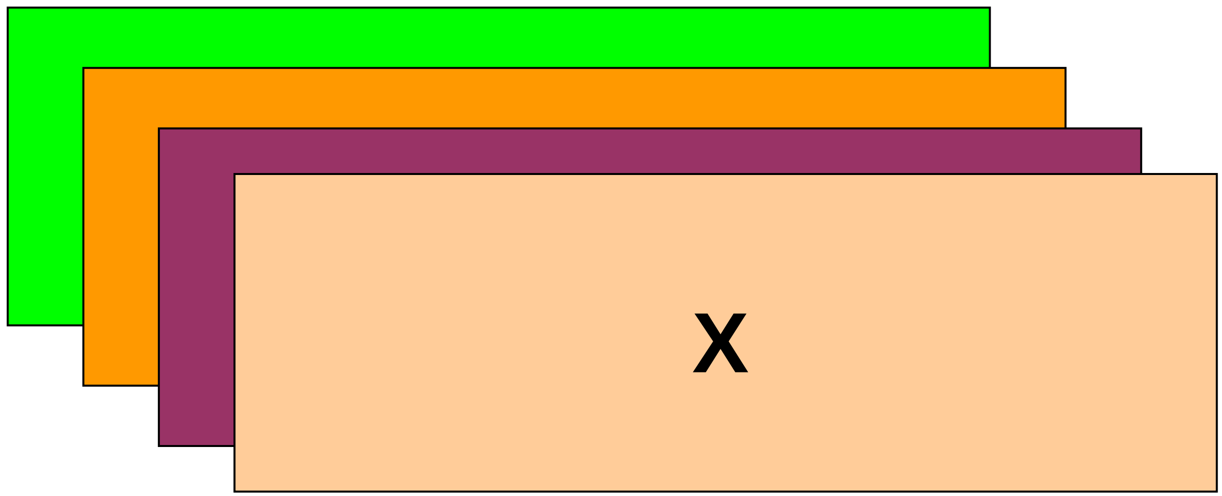
- Beck and Cohen (2003)
- Simple description of non-equilibrium systems

$$\boxed{e^{-\beta\mathcal{H}} \Rightarrow \int_0^\infty d\beta\, f(\beta) e^{-\beta\mathcal{H}}}$$

- Can be incorporated into Gaussian RMT

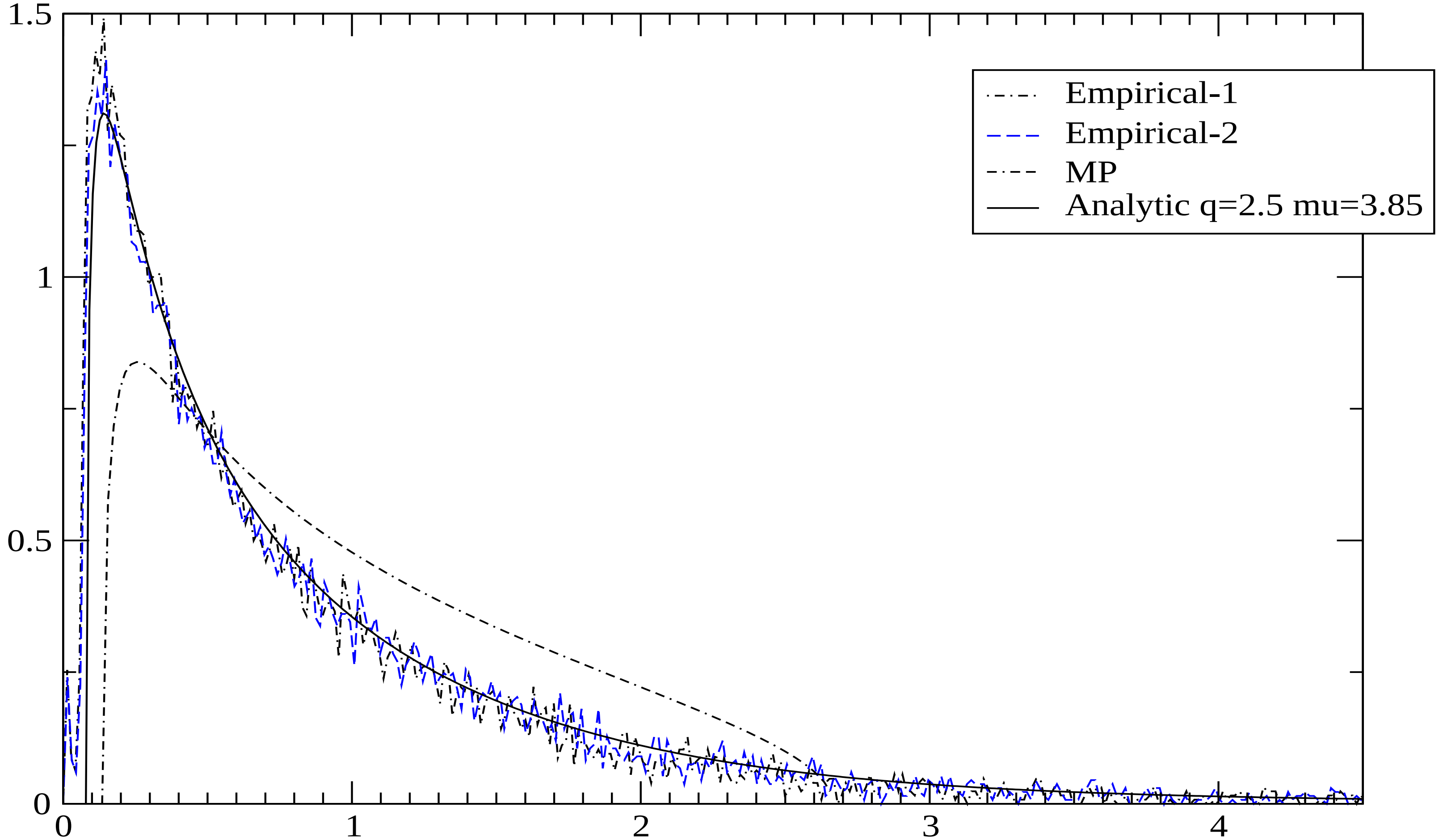
[A.Y. Abul-Magd, Phys. Rev. E **72**,066114 (2005)]

Superstatistical model



- The variance of $\mathbf{X}$-entries fluctuates from one sample to another
- The spectral density and all correlation functions of $\mathbf{W} = \mathbf{X}\mathbf{X}^\dagger$ are modified
- The model is exactly solvable

[Biroli *et al.,* Acta Phys. Pol. **38**, 4009 (2007)]



Three Superstatistical Classes

- χ^2 -distribution

$$\boxed{f_1(\eta) \propto \eta^{\gamma-1} \exp\left(-\gamma\eta\right)}$$

- Inverse χ^2 -distribution

$$\boxed{f_2(\eta) \propto \frac{1}{\eta^{\gamma+2}} \exp\left(-\gamma/\eta\right)}$$

- Log-Normal Distribution

$$\boxed{\cancel{f_3(\eta) \propto \frac{1}{\eta} \exp\left(-[\log(\eta/\mu)]^2\right)}}$$

$$\boxed{P_2\left(\mathbf{X}\right) \to \exp\left[-\eta\beta\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right]}$$

$$\boxed{P_2\left(\mathbf{X}\right) \to \frac{1}{\eta^{\gamma+2}}\exp\left[-\frac{\gamma}{\eta}\right]\exp\left[-\eta\beta\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right]}$$

$$\boxed{P_2\left(\mathbf{X}\right) \to \int_0^\infty d\xi\frac{1}{\xi^{\gamma+2}}\exp\left(-\frac{1}{\xi}\right)\exp\left[-\xi\beta\gamma\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right]}$$

$$\boxed{P_2\left(\mathbf{X}\right) \equiv \int_0^\infty d\xi\frac{1}{\xi^{\gamma+2-(\beta/2)MN}}\exp\left(-\frac{1}{\xi}\right)\frac{\exp\left[-\xi\gamma\beta\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right]}{Z(\xi)}}$$

$$\propto\left(\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right)^{\frac{1}{2}(\gamma+1-\beta NM/2)}K_{\gamma+1-\beta NM/2}\left(2\sqrt{\beta\gamma\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)}\right)$$

$$\boxed{\gamma\to\infty \qquad\qquad P_2(\mathbf{X})\to P_{WL}(\mathbf{X})}$$

Two distributions of matrix elements

$$\boxed{P_1(\mathbf{X}) \propto \left(1 + \frac{\beta}{\gamma}\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right)^{-\gamma+\frac{1}{2}\beta NM}.}$$

Power-law decay of spectral correlations

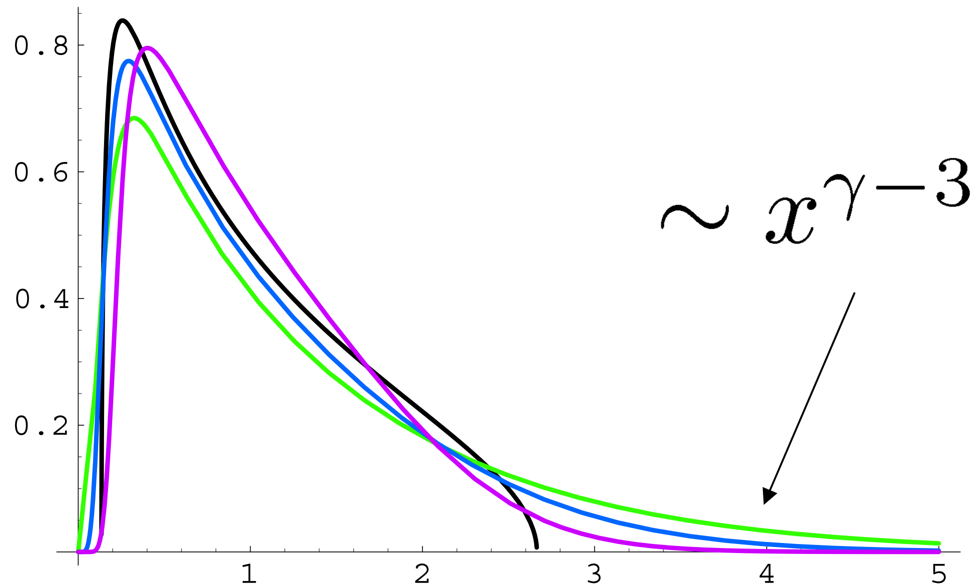
$$\boxed{P_2(\mathbf{X}) \propto \left(\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)\right)^{\frac{1}{2}(\gamma+1-\beta NM/2)} K_{\gamma+1-\beta NM/2}\left(2\sqrt{\beta\gamma\mathrm{Tr}(\mathbf{X}\mathbf{X}^\dagger)}\right)}$$

Exponential decay of spectral correlations

Both are obtained as averages of the standard Wishart-Laguerre weight over different distributions of the variance of **X**

$$\mathcal{P}_\gamma(\lambda_1,...,\lambda_N) \propto \int_0^\infty d\xi \frac{1}{\xi^{\gamma+2-\frac{\beta}{2}NM}} \exp\left[-\frac{1}{\xi}\right] \mathcal{P}_{WL}(\lambda_1,...,\lambda_N;\xi)$$

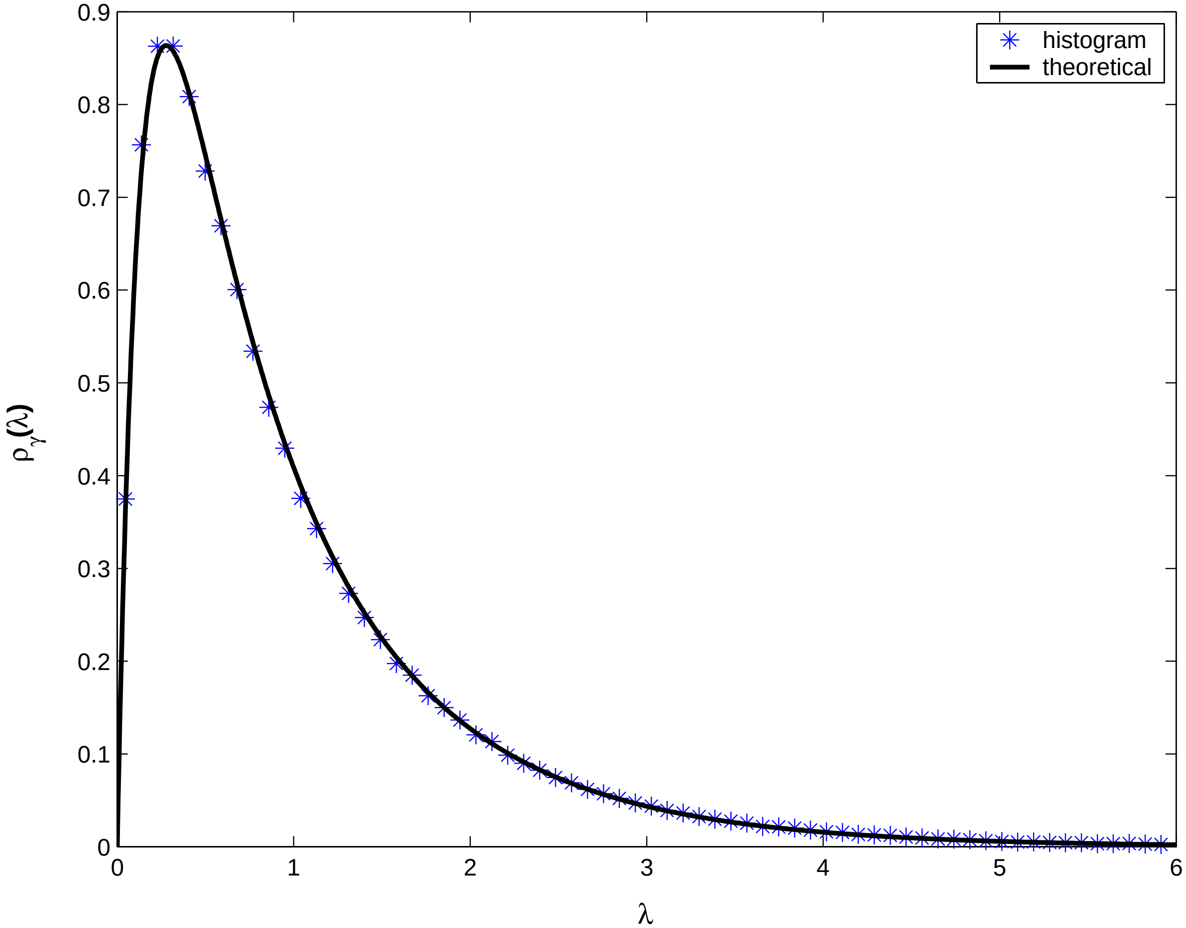
$$\mathcal{P}_{WL}(\lambda_1,...,\lambda_N;\xi) \propto \prod_{i<j}^{N} |\lambda_j-\lambda_i|^\beta \prod_{k=1}^{N} \lambda_k^{\frac{\beta}{2}(M-N+1)-1} \exp\left(-\xi\gamma\beta\sum_{\ell=1}^{N}\lambda_\ell\right).$$



$$\sim x^{\gamma-3/2}e^{-\kappa x}$$

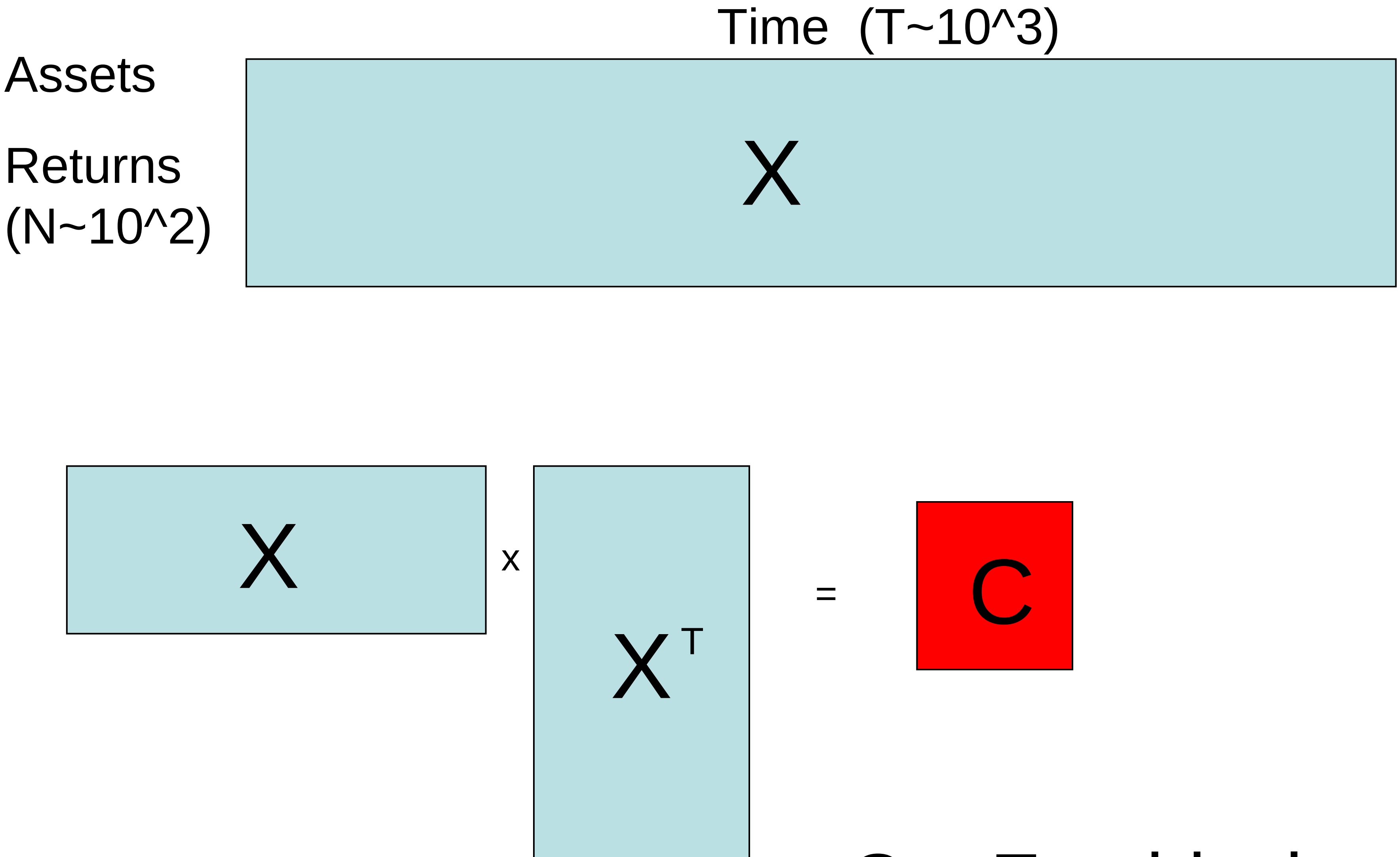
$$\rho_\gamma(x) \propto x^\gamma \int_{X_-}^{X_+} dt\ \exp\left[-\frac{x(\gamma+1)}{tc}\right] t^{-\gamma-2}\sqrt{(t-X_-)(X_+-t)}$$

Numerical Simulations



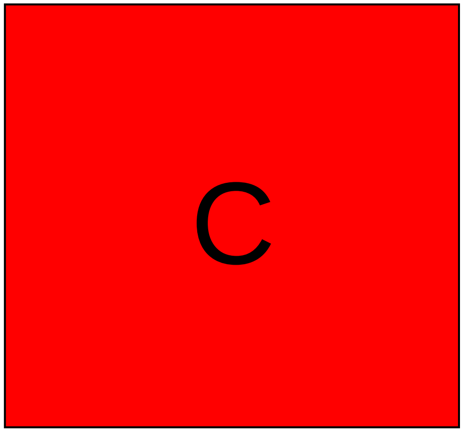
Conclusions

- Random Covariance Matrices
- Variance of data fluctuates from one sample to another according to a normalized distribution $f(\eta)$
- Integral Transform of Wishart-Laguerre ensembles, depending on a single deformation parameter γ
- The model can be solved exactly
- Expected applications beyond the usual superstatistical classes



C = Empirical Covariance Matrix

Empirical Covariance Matrix



- N x N
- Real
- Symmetric
- Positive definite

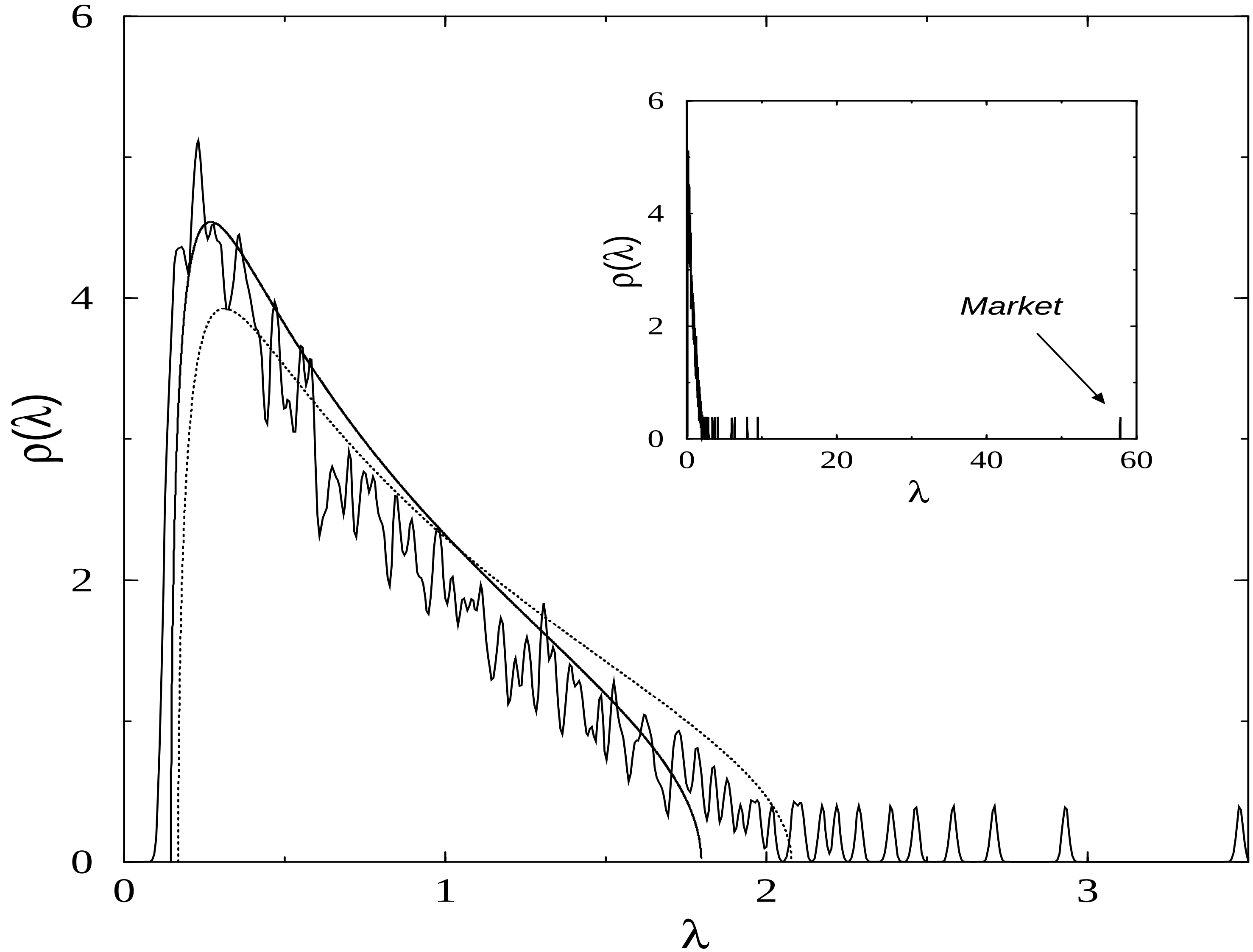
Eigenvalues are real and positive

What is the amount of randomness in financial data?

↓

Random Covariance Matrices: Wishart-Laguerre ensemble

[Laloux *et al,* Phys. Rev. Lett. 83 (1999) 1467]



A new model of random covariance matrices

- Exactly solvable
- Recovers Wishart-Laguerre in a certain limit
- Power-law tails

$$P(\mathbf{X}^T\mathbf{X}) \propto \exp\left(-\,\mathrm{Tr}\left(\mathbf{X}^T\mathbf{X}\right)\right)\det{}^{\nu}\left(\mathbf{X}^T\mathbf{X}\right)$$

$$\gamma \to \infty$$

$$P_\gamma(\mathbf{X}^T\mathbf{X}) \propto \left(1+\frac{1}{\gamma}\mathrm{Tr}\left(\mathbf{X}^T\mathbf{X}\right)\right)^{-\gamma} \det{}^{\nu}\left(\mathbf{X}^T\mathbf{X}\right)$$

Salient features of the deformed model

- The data matrix **X** has entries correlated in an intricate way.
- It recovers Wishart-Laguerre in the limit of large γ.
- We can hope to get power-law tails.
- It remains to prove that it is exactly solvable!

Gamma-integral Identity

$$(1+\mathrm{Y})^{-\gamma}=\frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-\xi}\xi^{\gamma-1}e^{-\xi\mathrm{Y}}\,\mathrm{d}\xi$$

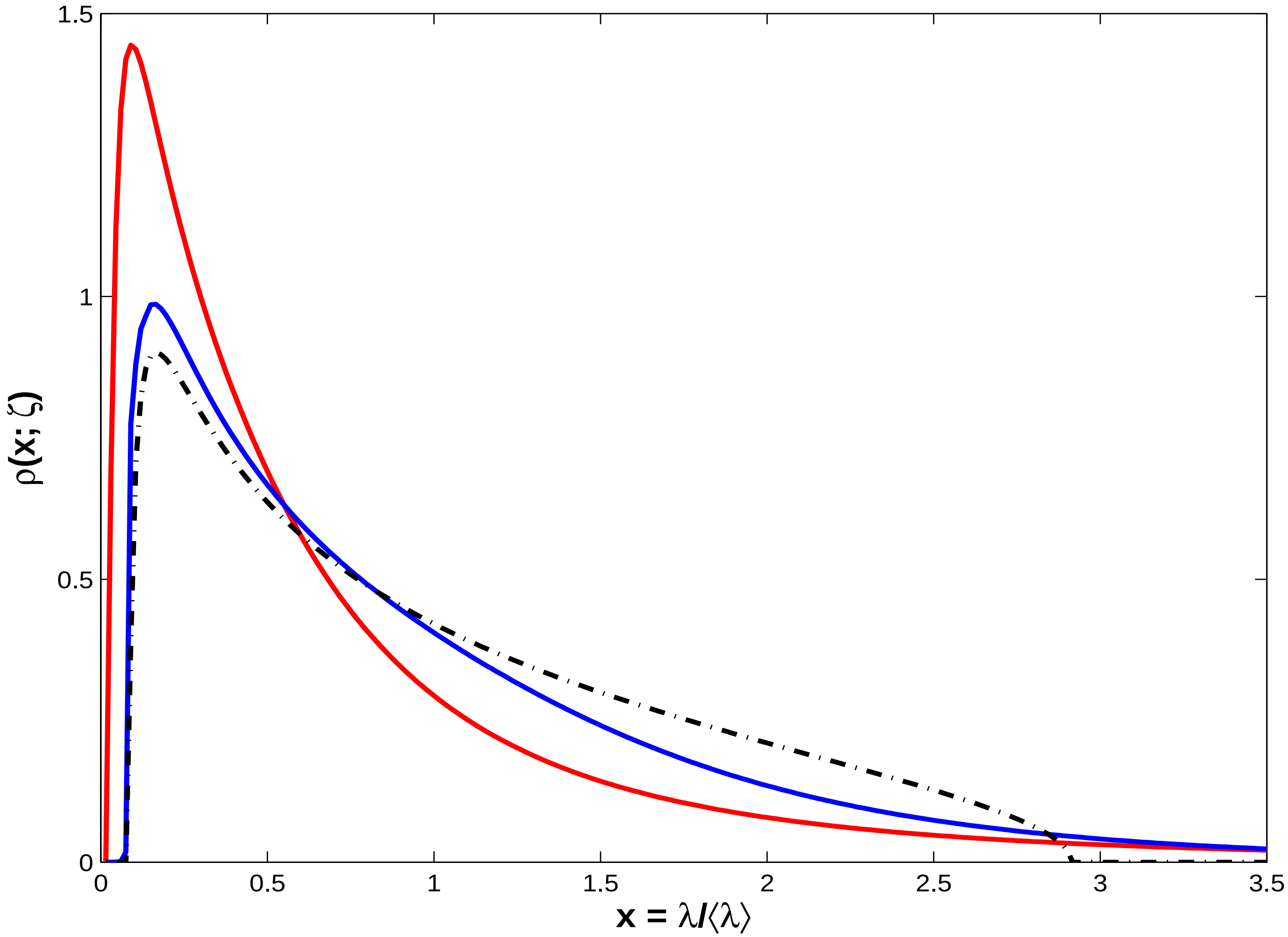
$$P_\gamma(\mathbf{X}^T\mathbf{X})\propto\left(\underbrace{1+\frac{1}{\gamma}\mathrm{Tr}\left(\mathbf{X}^T\mathbf{X}\right)}_{\mathrm{Y}}\right)^{-\gamma}\det{}^{\nu}\left(\mathbf{X}^T\mathbf{X}\right)$$

The probability measure of the generalized model can be represented as a gamma-integral of the Wishart-Laguerre measure!

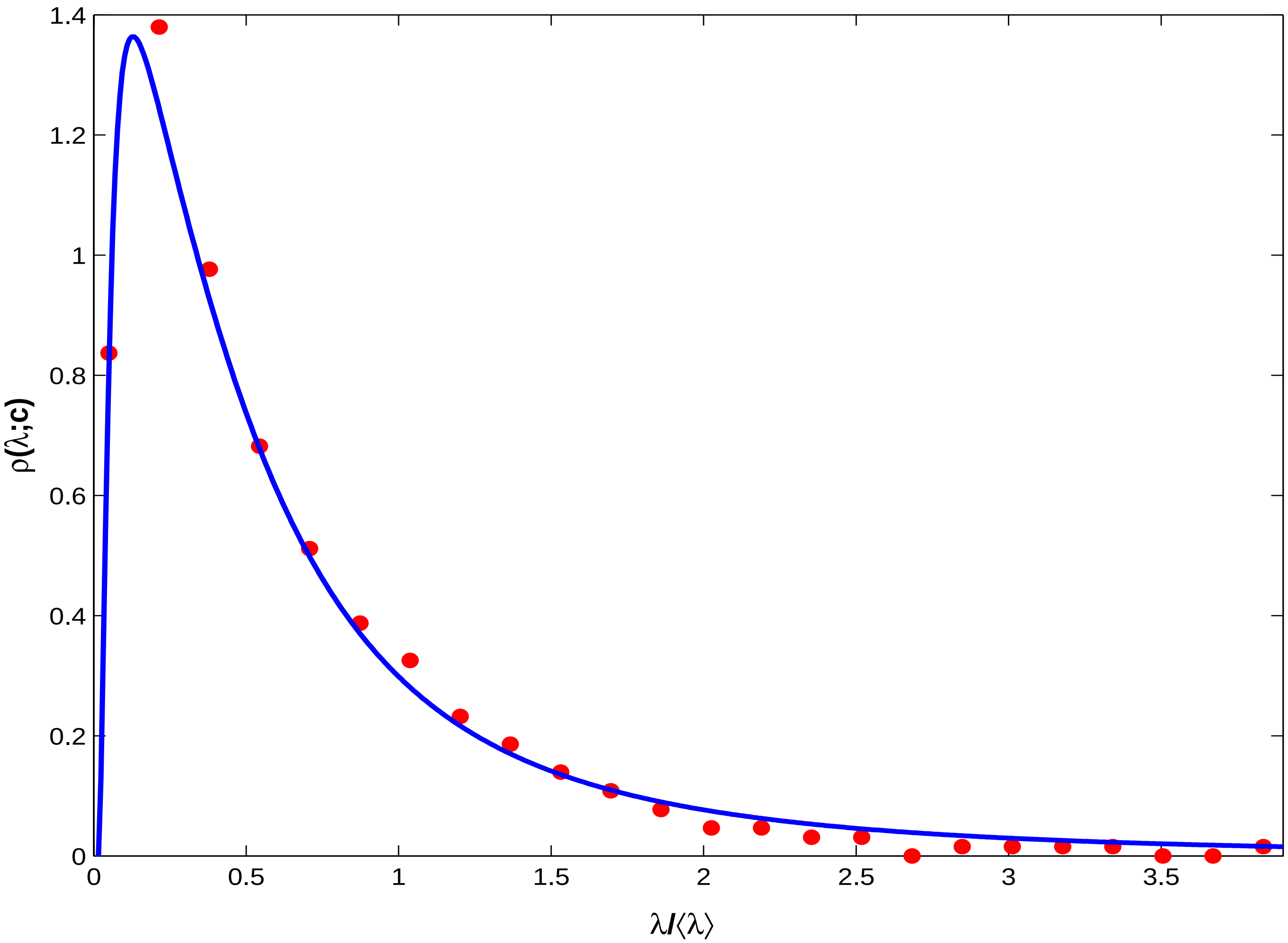
Exact results

- Density of eigenvalues for finite N and γ
- Macroscopic density of eigenvalues in a certain double scaling limit
- Generalization of the Wigner's surmise for the gap distribution in a 2x2 model
- Distribution of the smallest eigenvalue

Density of Eigenvalues



Comparison to Financial Data



Related works

- Z. Burda *et al.*, Phys. Rev. E **74**, 041129 (2006).

- A.C. Bertuola *et al.*, Phys. Rev. E **70**, 065102 (2004) .

- G. Biroli *et al.*, Acta Phys. Pol. **38**, 4009 (2007).

Summary

- Exactly solvable deformation of Wishart-Laguerre ensemble of random matrices.
- Only one free parameter ζ, such that we recover WL for $\zeta \to \infty$
- Good agreement with eigenvalue distribution from financial data

Outline

- Covariance matrices and financial data.
- Random covariance matrices:
  the Wishart-Laguerre ensemble
- A simple deformation: power-law tails
- Comparison with financial data