On the mean density of complex eigenvalues for an ensemble of random matrices with prescribed singular values

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Brunel University, Dec. 2008

Outline

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1. Introductions
   - A Question by H. Weyl
   - Quantum Chaotic Scattering
2. My Work
   - Methods and Results
   - Examples
3. Summary and Outlook

Preliminaries

- Mean Density Function: $\{z_j\}$ Eigenvalues of A

$$\langle \sharp z \in D \rangle = N \int_{D \subset \mathbb{C}^2} \rho(z) \, \mathrm{d}^2 z$$

$$\rho(z) = \langle \frac{1}{N} \sum_j \delta^{(2)}(z - z_j) \rangle_A$$

- Singular Values:
  - $N \times N$ Complex Matrix $A = U\sqrt{G}V$
  - $\sqrt{G} = \mathrm{diag}(\sqrt{g_1}, \ldots, \sqrt{g_N}) \geq 0$ Eigenvalues of $\sqrt{AA^\dagger}$

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Weyl's Question

- Weyl (1949) and Horn (1954)
  Complex Matrix A with Prescribed Singular Values $\{\sqrt{g_i}\}$,
  Eigenvalues of A?

Numerical Simulations

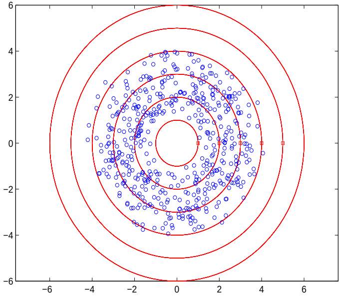

Figure: 100 samples of matrices with $\sqrt{G} = \mathrm{diag}(1, 2, 3, 4, 5)$

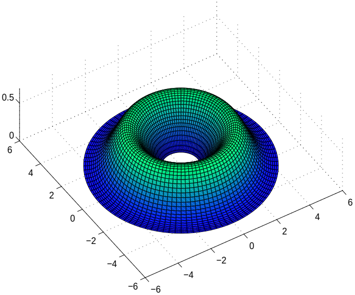

Figure: Density Function in 3D

Histogram

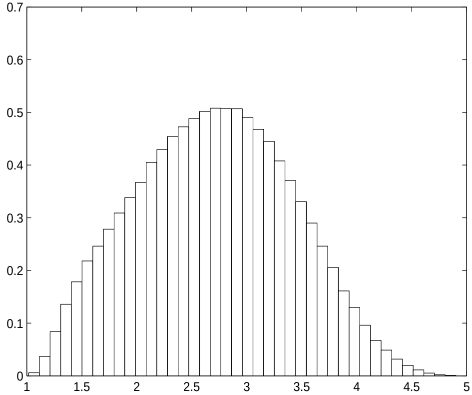

Figure: 5×5 matrices A with singular value $\sqrt{G} = \mathrm{diag}(1, 2, 3, 4, 5)$

Outline

- A System rep. by Chaotic Map embedded in Outer World Chaotic Region with M Open Channels
- $\hat{A} = \hat{U}\sqrt{1 - \hat{\tau}\hat{\tau}^{\dagger}}$, $\hat{\tau}$: $N \times M$, $M \leq N$, $\tau_{ij} = \delta_{ij}\tau_j$

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Methodology

- $A = U\sqrt{G}V$
- Starting Formula, Y. Fyodorov and H-J. Sommers, 97

$$\rho(z) = -\frac{1}{\pi}\lim_{\kappa\to 0}\frac{\partial}{\partial \bar{z}}\lim_{z_b\to z}\frac{\partial}{\partial z_b}\left\langle \frac{\det\begin{pmatrix}\kappa & i(z-A)\\ i(\bar{z}-A^\dagger) & \kappa\end{pmatrix}}{\det\begin{pmatrix}\kappa & i(z_b-A)\\ i(\bar{z}_b-A^\dagger) & \kappa\end{pmatrix}}\right\rangle_U$$

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SUSY Colour-Flavour Transformation

- Gaussian Integral over Graded-Vectors
- Supersymmetric Colour-Flavour Transformation

$$\int dU \exp i \left[\bar{\psi}_1^i A_{ij} \psi_2^j + \bar{\psi}_2^i A_{ij}^\dagger \psi_1^j\right] = \int D(Q,\tilde{Q}) \; \exp i \left[\bar{\psi}_1^i Q \psi_1^i + g_i \bar{\psi}_2^i \tilde{Q} \psi_2^i\right]$$

2×2 graded matrices Q, AIII|AIII

$$Q = \begin{pmatrix} q_b & \eta_1 \\ \eta_2 & q_f \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} \bar{q}_b & \sigma_1 \\ \sigma_2 & -\bar{q}_f \end{pmatrix}$$

$q_b \in \mathrm{U}(1,1)/\mathrm{U}(1) \times \mathrm{U}(1) = \mathrm{H}^2$ and $q_f \in \mathrm{U}(2)/\mathrm{U}(1) \times \mathrm{U}(1) = \mathrm{S}^2$

- Analytic Contintuation

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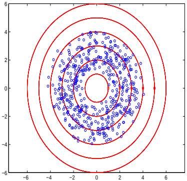
A Statistical Answer



- Averaged Eigenvalue Density Function

$$\rho(|z|^2) = \begin{cases} 0 & |z|^2 < g_1 < g_2 \cdots < g_N \\ \frac{1}{N} \sum_{i=k+1}^{N} \mathrm{F}_\Delta(g_i) & g_1 < \cdots < g_k < |z|^2 < g_{k+1} < \cdots < g_N \\ 0 & g_1 < g_2 \cdots < g_N < |z|^2 \end{cases}$$

Definition of Notation

$\mathrm{F}_\Delta(g_i)$ a Symmetric Function of $G\backslash\ g_i$

$$\mathrm{F}_\Delta(g_i) = \frac{1}{\prod_{j=1}^{N}{}' (g_i - g_j)} (g_i - |z|^2)^{N-2} \sum_{l=0}^{N-1} \mathrm{s}_{[i]}^{l} |z|^{-2(l+1)} \frac{1}{C_{N-1}^{l}} \left[l g_i + (N-1-l)|z|^2 \right]$$

$$= \frac{(g_i - |z|^2)^{N-2}}{\prod_{j=1}^{N}{}' (g_i - g_j)} \int_0^\infty \frac{N dt}{(1+t)^{N+2}} \det\left(1 + \frac{t}{|z|^2} G_{[i]}\right) \left[N - t + \frac{g_i}{|z|^2}(Nt - 1) \right]$$

Degeneracies

- When G Has Degeneracies

$$G = (\ldots, [g_{k_1}, \ldots, g_{k_1+i_1}], \ldots, [g_{k_s}, \ldots, g_{k_s+i_s}], \ldots)$$

- $$\mathrm{f}_n^{[k,i]}(g) \stackrel{\text{def}}{=} \frac{(g-|z|^2)^{N-2}}{\prod_{j=1}^{k-1}\prod_{j=k+i+1}^{N}(g-g_j)} \sum_{l=n}^{N-1} \mathrm{s}^{l-n}_{[k,\ldots,k+n]} \frac{|z|^{-2(l+1)}}{C^l_{N-1}} \left[lg + (N-1-l)|z|^2\right]$$

$$\mathrm{F}_{\Delta}^{[k,i]} \stackrel{\text{def}}{=} \sum_{n=0}^{i} \frac{(-)^n}{(i-n)!} \frac{d^{i-n}}{dg_{k+n}^{i-n}} \mathrm{f}_n^{[k,i]}(g_{k+n})$$

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Check Special Cases

- Truncated Unitary Matrix: $A = U\sqrt{G}$, $G = \mathrm{diag}(0I_M, I_{N-M})$

$$\rho(|z|^2) \propto (1-|z|^2)^{M-1}\left(\frac{d}{d|z|^2}\right)^M \frac{1-|z|^{2N}}{1-|z|^2}$$

K. Życzkowski and H-J Sommers 2000

- Rank-1 Deviation from Unitary: $A = U\sqrt{G}$, $G = \mathrm{diag}(g_1, gI_{N-1})$

$$\rho(|z|^2) = \frac{(|z|^2-g_1)^{N-2}}{(g-g_1)^{N-1}|z|^{2N}}\Bigg((N-1)(|z|^{2N}+g^{N-1}g_1) + \sum_{k=0}^{N-2}\left[(N-2-k)g+kg_1\right]g^k|z|^{2(N-1-k)}\Bigg)$$

Y. Fyodorov 2001

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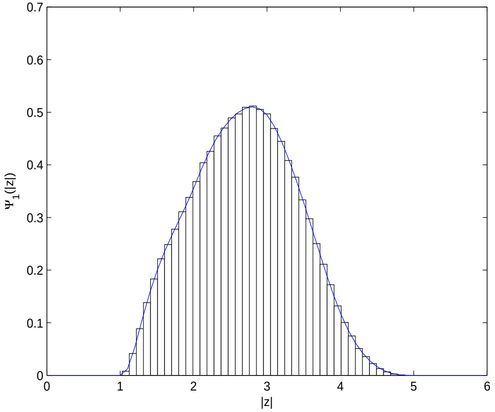
Compare with Numerics



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Summary and Outlook

- Reference, YW and Y. Fyodorov, JPA 08.
- Large N limit?
- Phase Transition?
- $U \in$COE, Mean Density Function of $U\sqrt{G}$?

Thank You for Your Attention!