RANDOM-MATRIX APPROACH TO RPA EQUATIONS

X. Barillier-Pertuisel
IPN, Orsay

Oriol Bohigas
LPTMS Orsay

Hans A. Weidenmüller
MPIK, Heidelberg

arXiv nucl.th. 0807.35
Ann. Phys. (NY)

Random Matrices
Brunel University, 18-19 Dec. 2009

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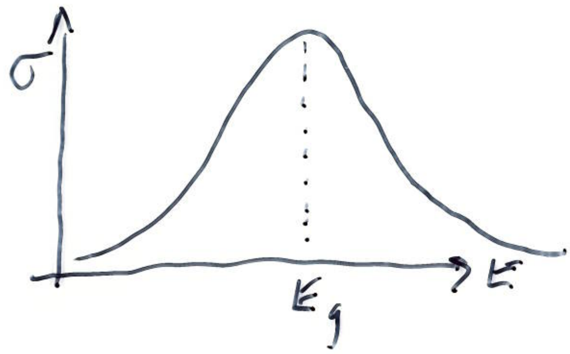
Prototype of collective state in nuclei:

nuclear photoeffect



total photoabsorption cross-section

dominated by $\sigma(\gamma, n)$

$$E_g \sim 80\, A^{-1/3} \quad \text{MeV}$$

width ~ 5–6 MeV

$$m_1 \sim \int \sigma(E)\, dE \sim \sum (E_n - E_0)\, |\langle n | D | 0 \rangle|^2$$

$$\sim \langle 0 | [D, [H, D]] | 0 \rangle$$

Thomas Reiche Kuhn sum rule

$$m_0 \sim \int \frac{\sigma(E)}{E}\, dE \sim \sum_n |D_{n0}|^2$$

$$m_{-1} \sim \int \frac{\sigma(E)}{E^2}\, dE \sim \sum \frac{|D_{n0}|^2}{E_n - E_0}$$

Bremstrahlung weighted crosssection

nuclear (dipole) polarizability

In h.o. picture $E_g \sim 2\hbar\omega$ for dipole (and not $1\hbar\omega$)

<u>Upward collective shift</u>

1 MOTIVATIONS.

Random-phase approximation (RPA) is a standard tool of many-body physics. First used in condensed-matter physics but later also in other areas and especially in nuclear-structure physics, in particular to describe collective states

Hamiltonian in the Hartree-Fock representation

$$H = H_0 + V_{res}$$

Lowest excited states of H_0, the particle-hole excitations, are coupled to one another by the residual interaction

Take for the residual interaction a separable interaction (Brown, Evans, Thouless)

$$V_{res} = -\frac{1}{2} \chi \sum_{ij} \hat{Q}(i) \cdot \hat{Q}^*(j)$$

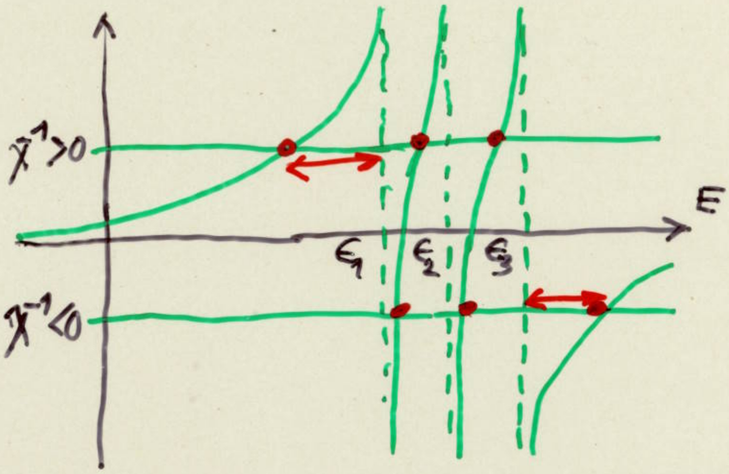
Leads to extreme collectivity

– Tamm-Dancoff approximation

Diagonalize H in the particle-hole subspace.

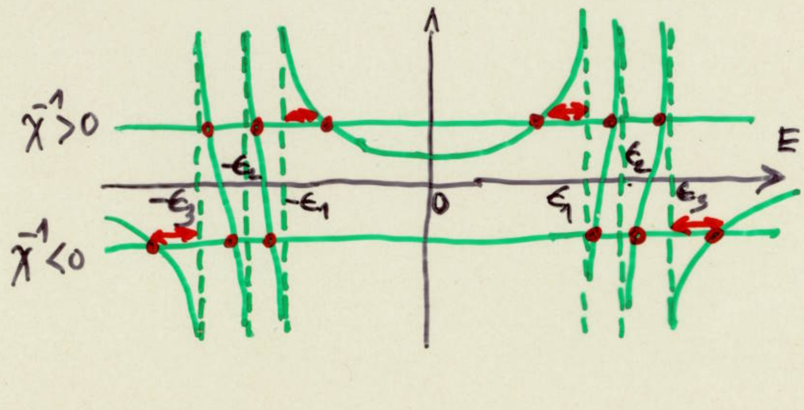
Leads to the dispersion equation for the excitation energies E

$$\sum_{mi} \frac{|Q_{mi}|^2}{\epsilon_{mi} - E} = \frac{1}{\chi}$$



– RPA approximation. Leads to dispersion equation for the excitation energies

$$2\sum_{mi} \frac{|Q_{mi}|^2 \epsilon_{mi}}{\epsilon_{mi}^2 - E^2} = \frac{1}{\chi}$$



- Maximum departure from schematic model (from extreme collectivity): take matrix elements randomly. Yields unitarily invariant random matrix model

- Study spectrum using (generalized) Pastur equation as a function of coupling between states at positive and negative energies

- When does the spectrum become instable (i.e. when do eigenvalues become complex)? Distribution of eigenvalues in complex energy plane?

2. RPA EQUATIONS.

$$H = \sum_{k,l} A^{0}_{k,l} b^{+}_{k} b_{l} + \frac{1}{2}\sum_{kl} C_{kl} b^{+}_{k} b^{+}_{l} + \frac{1}{2}\sum_{kl} C^{*}_{kl} b_{k} b_{l}$$

b^{+}, b : bosonic operators $\quad [b_k, b_l^{+}] = \delta_{kl}$

N-dimensional space for particle-hole pairs.

RPA equations have dimension 2N

$$\vec{X}^{\nu} = (X^{\nu}_{k}, Y^{\nu}_{k})$$

$$\mathcal{H}^{0} \vec{X}^{\nu\,T} = (E_{\nu} - E_{0}) \vec{X}^{\nu\,T}$$

$$\mathcal{H}^{0} = \begin{pmatrix} A^{0} & C \\ -C^{*} & -(A^{0})^{*} \end{pmatrix}$$

Matrix A^{0} : hermitean

C : symmetric

$\therefore$ $\mathcal{H}^{0}$: not hermitean

Eigenvalues not necessarily real.

Non real eigenvalues : instability

Symmetries

$$\mathcal{M} = \begin{pmatrix} 0 & \mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix} \qquad \mathcal{M}' = \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_N \end{pmatrix}$$

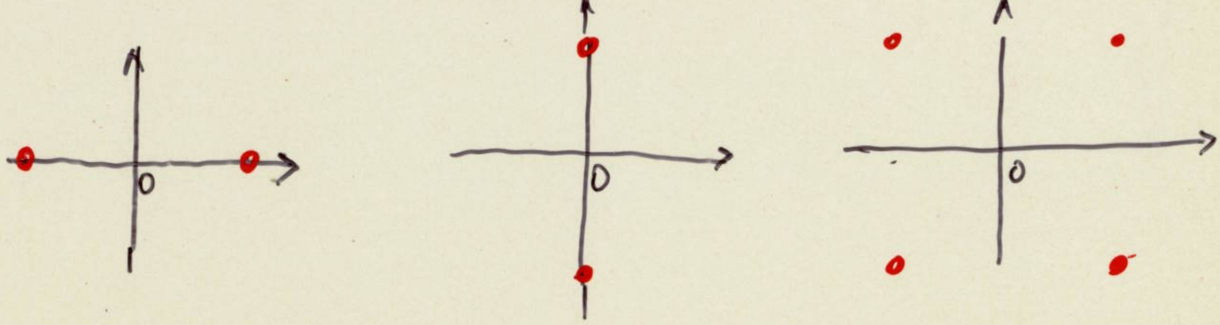
$$\mathcal{M}\,(\mathcal{H}^0)^*\,\mathcal{M} = -\mathcal{H}^0$$

If λ is eigenvalue, $-\lambda^*$ also is

$$\mathcal{M}'\,(\mathcal{H}^0)^\dagger\,\mathcal{M}' = \mathcal{H}^0$$

If λ is eigenvalue, λ^* also is

$\Rightarrow$ Real and purely imaginary eigenvalues come in pairs with opposite signs. Complex eigenvalues with non-vanishing real and imaginary parts come in quartets arranged symmetrically with respect to real and imaginary axis



Hermitean matrix A^0 causes repulsion amongst positive eigenvalues, matrix $-(A^0)^*$ causes repulsion amongst negative eigenvalues. For $C=0$, all eigenvalues are real.

Role of C

Eliminate negative energy subspace

Produce additional level repulsion amongst positive and amongst negative eigenvalues

Produce level attraction between positive and negative eigenvalues

With increasing strength of C, pairs of real eigenvalues coalesce at $E=0$ and then move along the imaginary axis in opposite directions

Instability of the RPA equations

3 RANDOM-MATRIX APPROACH

Degenerate particle-hole energies at r

$$A^{0} = r\,\mathbb{1}_N + A$$

A (particle-hole interaction): a GUE matrix

C a random matrix with i.g.v.

$$\langle A_{\mu\nu} A_{\rho\sigma} \rangle = \frac{\lambda^2}{N}\,\delta_{\mu\sigma}\,\delta_{\nu\rho}$$

$$\langle (\mathrm{Re}\, A_{\mu\nu})^2 \rangle = (1+\delta_{\mu\nu})\,\frac{\lambda^2}{2N}$$

$$\langle (\mathrm{Im}\, A_{\mu\nu})^2 \rangle = (1-\delta_{\mu\nu})\,\frac{\lambda^2}{2N}$$

$$\langle C_{\mu\nu} C^{*}_{\rho\sigma} \rangle = \frac{\gamma^2}{N}\,(\delta_{\mu\sigma}\,\delta_{\nu\rho} + \delta_{\mu\rho}\,\delta_{\nu\sigma})$$

$$\langle C_{\mu\nu} C_{\rho\sigma} \rangle = 0$$

$$\langle (\mathrm{Re}\, C_{\mu\nu})^2 \rangle = (1+\delta_{\mu\nu})\,\frac{\gamma^2}{2N}$$

$$\langle (\mathrm{Im}\, C_{\mu\nu})^2 \rangle = (1+\delta_{\mu\nu})\,\frac{\gamma^2}{2N}$$

A invariant under

$$A \to U A (U^{*})^{T}$$

C invariant under

$$C \to U C U^{T}$$

the RPA matrix becomes

$$\mathcal{H}^0 = \begin{pmatrix} r\mathbb{1}_N & 0 \\ 0 & -r\mathbb{1}_N \end{pmatrix} + \begin{pmatrix} \overset{\lambda^2}{A} & \overset{\gamma^2}{C} \\ \underset{\gamma^2}{-C^*} & \underset{\lambda^2}{-A^*} \end{pmatrix}$$

$$= \begin{pmatrix} r\mathbb{1}_N & 0 \\ 0 & -r\mathbb{1}_N \end{pmatrix} + \mathcal{H}$$

Generalized unitarity invariance:

Ensemble is invariant under

$$\mathcal{H} \to \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} \mathcal{H} \begin{pmatrix} U^+ & 0 \\ 0 & U^T \end{pmatrix}$$

The matrices are not Hermitian

We have also looked at the orthogonal case.

Aim: To study average spectrum for $N \to \infty$.

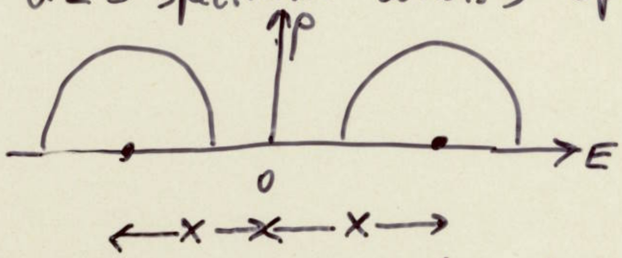
Model depends on two dimensionless parameters:

- $\alpha = \frac{\gamma^2}{\lambda^2}$

  which measures the relative strength of C compared to A;

- $x = \frac{r}{2\lambda}$

  for $\alpha = 0$ spectrum consists of two semicircles



x gives the distance of centers of semi-circles from origin at $E=0$

Spectral fluctuations have not been studied but are very likely to be the same as for GUE in each branch

Two limiting cases: $C=0$ and $A=0$

Question: Find value of α for which RPA equations become instable

To describe the single collective state

$$\mathcal{H}^0 = \begin{pmatrix} r\delta_{\mu\nu} + a_\mu a_\nu^* + A_{\mu\nu} & a_\mu a_\rho \; C_{\mu\rho} \\ -a_\sigma^* a_\nu^* - C_{\sigma\nu}^* & -r\delta_{\sigma\rho} - a_\sigma^* a_\rho - A_{\sigma\rho}^* \end{pmatrix}$$

$2N \times 2N$ und addition of four separable matrices

in the absence of A, C this gives the standard schematic model

Separability guarantees that the entire strength goes to collective state

Goal: compute the strength function

$$S_c(E) = \sum_j |\langle c|\psi_j\rangle|^2 \, \delta(E - E_j)$$

Separable matrix $a_\mu a_\nu^*$ has a single non-vanishing eigenvalue $|a|^2$

Variable coupling model

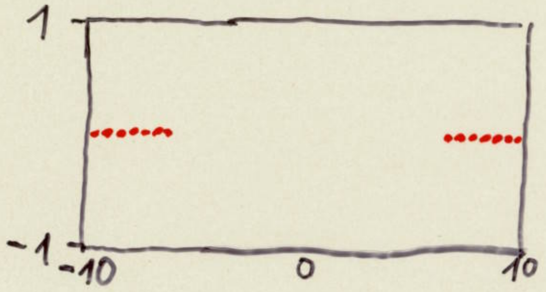
$(2N+2)\times(2N+2)$ matrix

$$
\left(\begin{array}{cccc:cccc}
r+|a|^2 & v & v & v & |a|^2 & w & w & w \\
v & & & & w & & & \\
v & & A_{\mu\nu} & & w & & C_{\mu\nu} & \\
v & & & & w & & & \\
\hdashline
-|a^*|^2 & -w^* & -w^* & -w^* & -r-|a|^2 & v & v & v \\
-w^* & & & & -v & & & \\
-w^* & & -C^* & & -v & & -A^* & \\
-w^* & & & & -v & & &
\end{array}\right)
$$

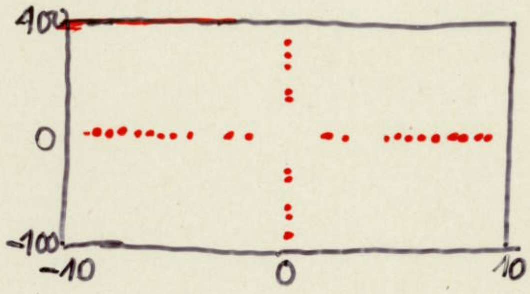
Distribution of eigenvalues in the complex energy plane

$N = 20 \quad , \quad x = 4$

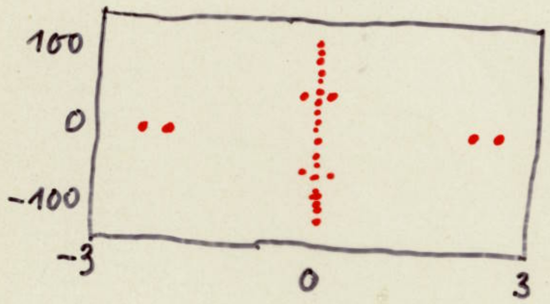
$\alpha = 1$



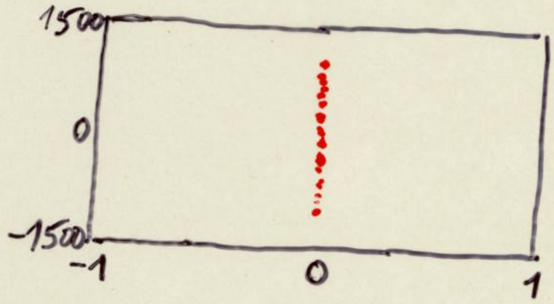
$\alpha = 30$

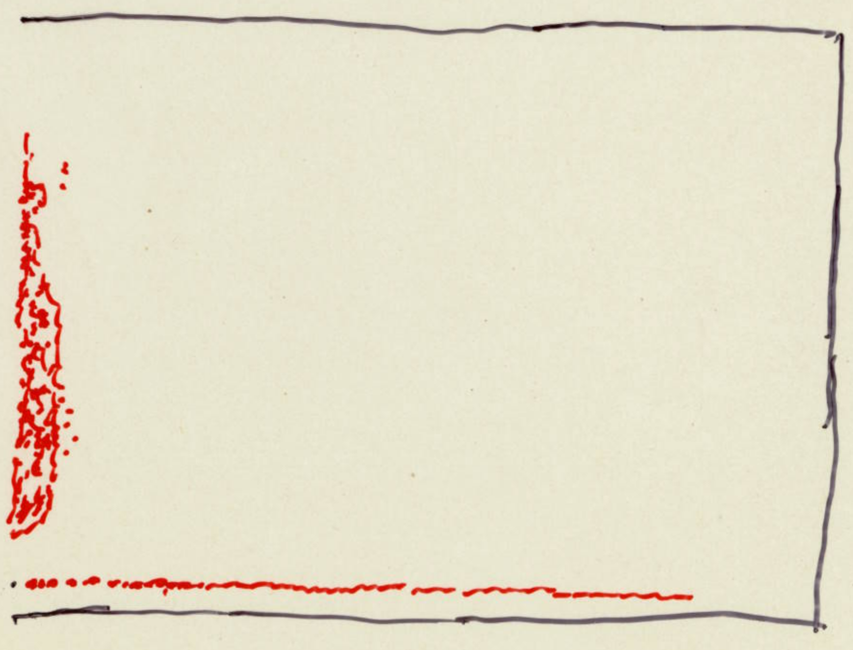


$\alpha = 50$



$\alpha = 150$





4 PASTUR EQUATION

Do this by calculating

$$\langle G(E) \rangle = \left\langle \left(E^+ \mathbb{1}_{2N} - \begin{pmatrix} r\mathbb{1}_N & 0 \\ 0 & -r\mathbb{1}_N \end{pmatrix} - \mathcal{H} \right)^{-1} \right\rangle$$

Imaginary part of average Green's function yields level density.

Expand in powers of $\mathcal{H}$

$$G(E) = G_0(E) + \sum_{n=1}^{\infty} G_0(E) \left(\mathcal{H} G_0(E) \right)^n$$

where

$$G_0(E) = \left(E^+ \mathbb{1}_{2N} - \begin{pmatrix} r\mathbb{1}_N & 0 \\ 0 & -r\mathbb{1}_N \end{pmatrix} \right)^{-1}$$

Average each term in sum separately. For N large, keep only nested contributions. Yields Pastur equation

$$\langle G(E) \rangle = G_0(E) + G_0(E) \langle \mathcal{H} \langle G(E) \rangle \mathcal{H} \rangle \langle G(E) \rangle$$

Define for $i=1,2$ the spectral density of subspace i in total spectrum (projection operators Q_i onto the two subspaces

$$\sigma_i(E) = \frac{\lambda}{N} \operatorname{Trace} Q_i \langle G(E) \rangle Q_i$$

and take trace of Pastur equation.

Yields two coupled equations for σ_1 and σ_2

$$\sigma_1 = \frac{\lambda}{E - r - \lambda(\sigma_1 - \alpha\sigma_2)}$$

$$\sigma_2 = \frac{\lambda}{E + r - \lambda(\sigma_2 - \alpha\sigma_1)}$$

For $\alpha = 0$ the equations are uncoupled and the imaginary parts of the solutions yield the two semi-circles of radius 2λ centered at r and at $-r$

5. Solutions. Numerical Results

To gain understanding of solutions, consider first $\alpha = 0$.
With $\varepsilon_i = (E - (-)^i r) / (2 \lambda)$, spectrum given by

$$\mathrm{Im}(\sigma_i) = \{1 - \varepsilon^2_i\}^{1/2} .$$

Usual semicircle law. Two branch points at $\varepsilon = \pm 1$. Each σ_i defined on Riemann surface with two sheets.

For $\alpha \neq 0$, solution defined on Riemann surface with four sheets. But which sheet to choose for physically relevant solution? Take α very small, use perturbation theory, find that we need pair (σ_1, σ_2) of solutions for which imaginary parts have opposite signs and $\mathrm{Im}(\sigma_1) > 0$. Then total level density $\rho(E)$ given by

$$\rho(E) = (N / (\pi \lambda)) \, \mathrm{Im} \, (\sigma_1 + \sigma_2) .$$

Eliminate σ_2 and obtain fourth-order equation for σ_1. Then σ_2 obtained from solution σ_1 via linear equation.

Pairs of solutions:

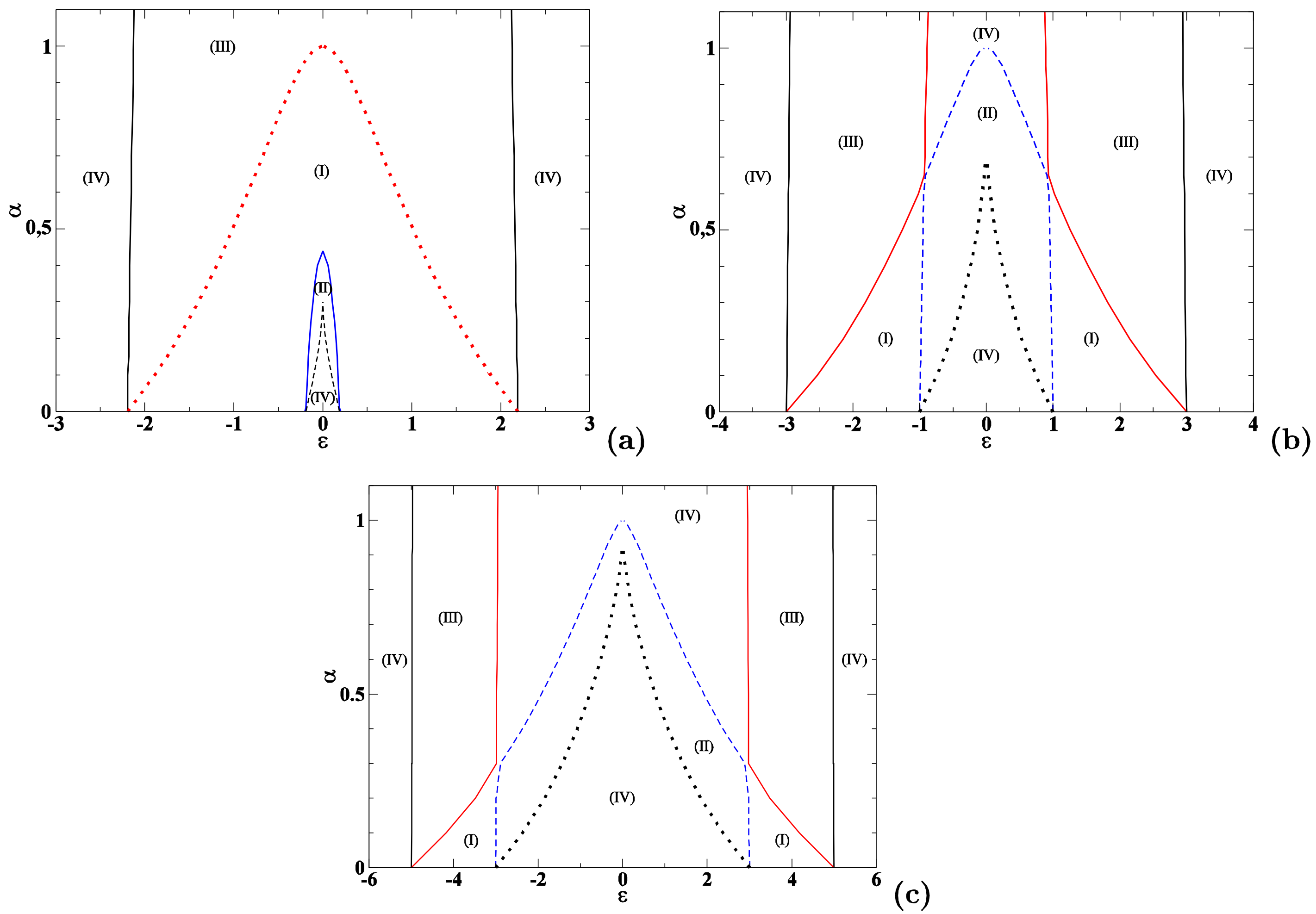
- Four pairs of complex solutions: Domain (I)

- Two pairs of real and two pairs of complex solutions with equal signs for Im(σ_1) and Im(σ_2): Domain (II)

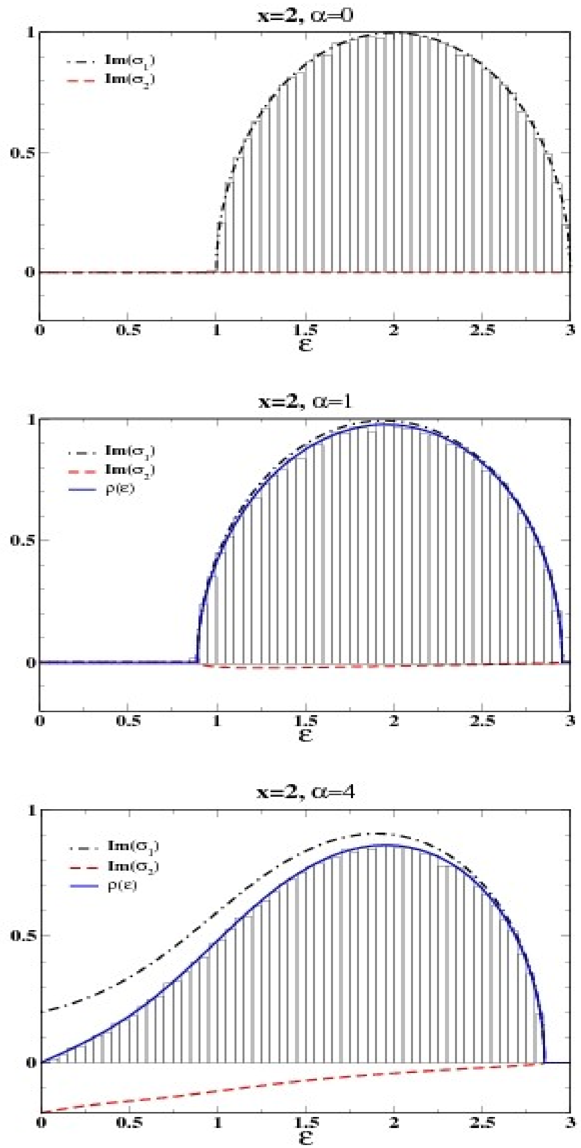
- Two pairs of real and two pairs of complex solutions with opposite signs for Im(σ1) and Im(σ2): Domain (III)

- Four pairs of real solutions: Domain (IV)

Physically interesting solutions only in domains (I) and (III).

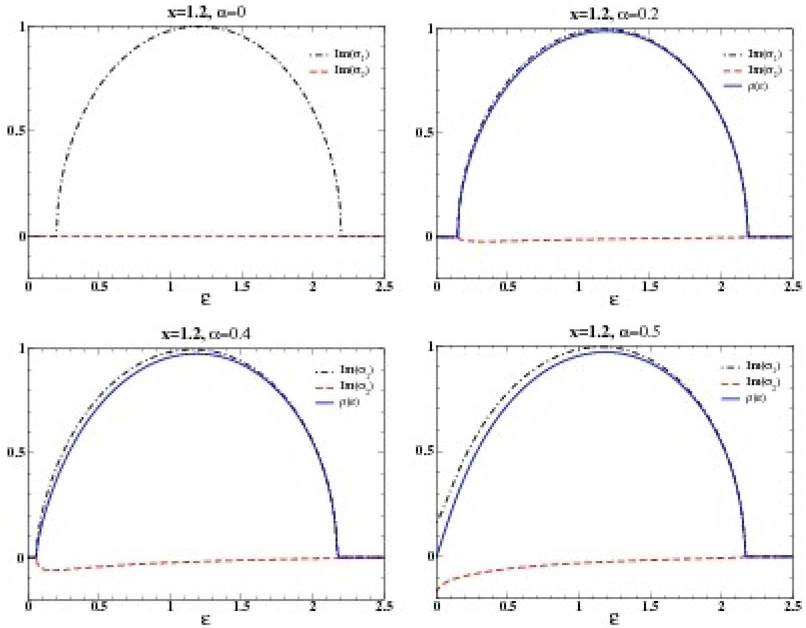


Case (a): x = 1.2   Case (b): x = 2.0   Case (c): x = 4.0



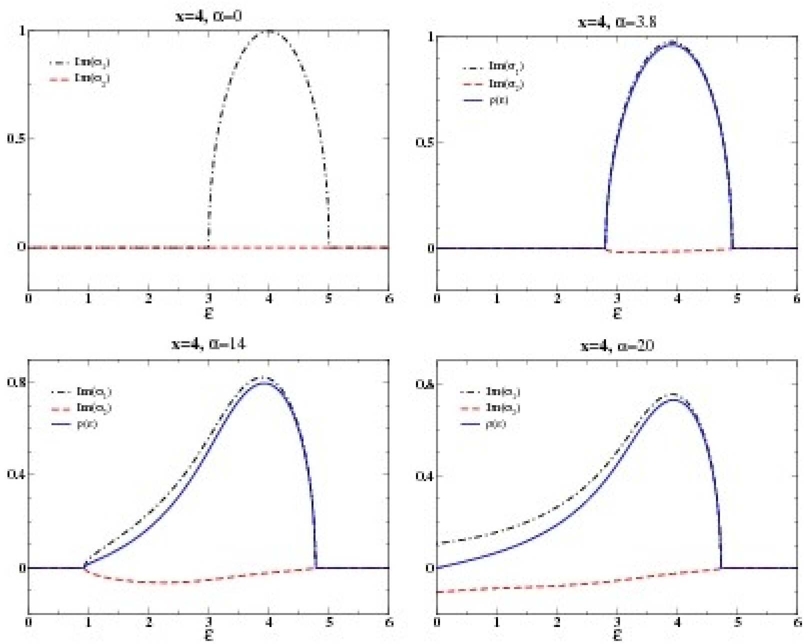
Comparison of results for Pastur equation with those of matrix diagonalization.

N = 50, 100 realizations



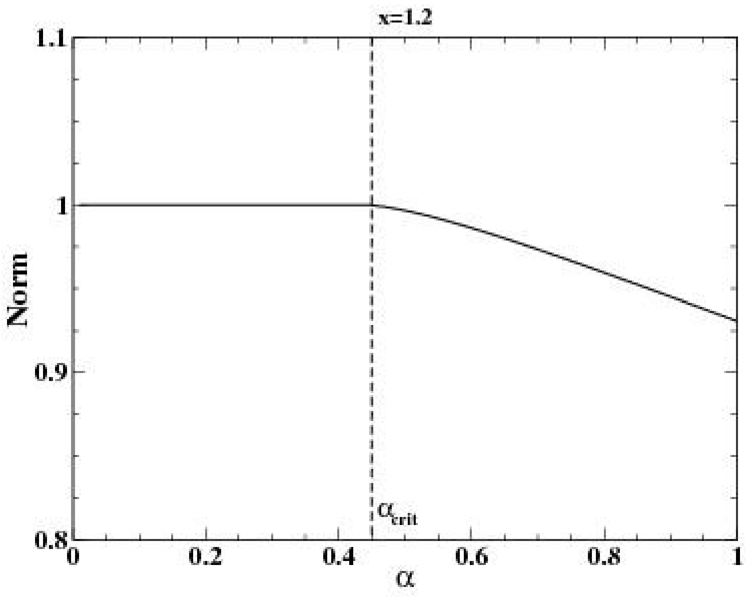
Evolution of two spectra with increasing α .

Upper panels: x = 1.2



Lower panels: x = 4

Normalization integral of total level density taken over real energies versus α. For α = 0, integral is normalized to unity.



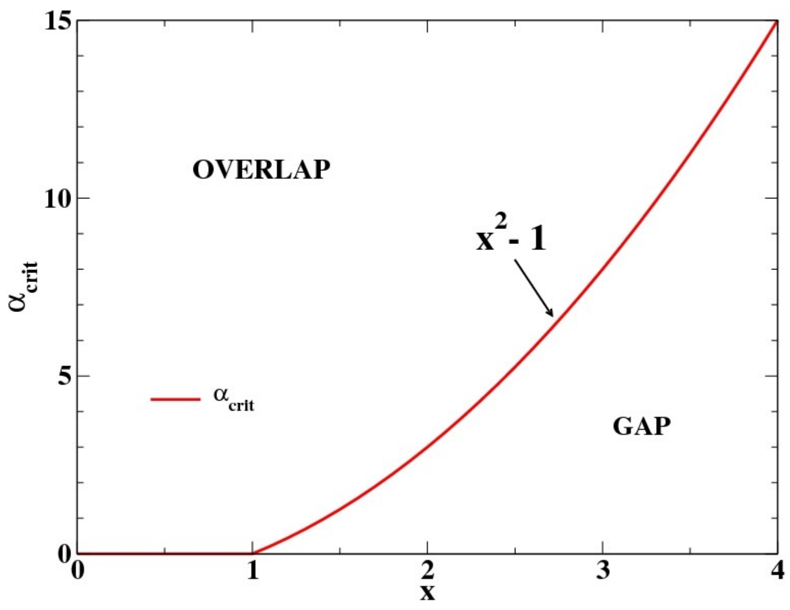
X = 1.2

6. Critical Strength

Find smallest value of α for which average spectra touch: Imaginary parts of a pair of physically acceptable solutions (σ_1, σ_2) have non-vanishing values at $E = 0$.

Determine α_{crit} analytically from solutions of fourth-order equation for σ_1 at $E = 0$. Find

$$\alpha_{crit} = x^2 - 1.$$



7. Summary

Random-Matrix model for RPA equations with “Generalized unitary invariance”.

Level repulsion between levels of same sign, level attraction between levels with opposite sign. Latter causes coalescence of pairs of eigenvalues with opposite signs at $E = 0$ and instability of RPA equations.

Use Pastur equation to derive two coupled equations for σ_1, σ_2. Surprisingly simple structure. Criterion for physically relevant solutions. Get average level density from (σ_1, σ_2). Two parameters: x and α.

With increasing α, semicircles are deformed and move toward each other. RPA instability for average spectrum: The deformed spectra touch. Critical strength $\alpha_{crit} = x^2 - 1$.