The $\mathcal{O}(\mathfrak{n})$ model on random lattices of all topologies

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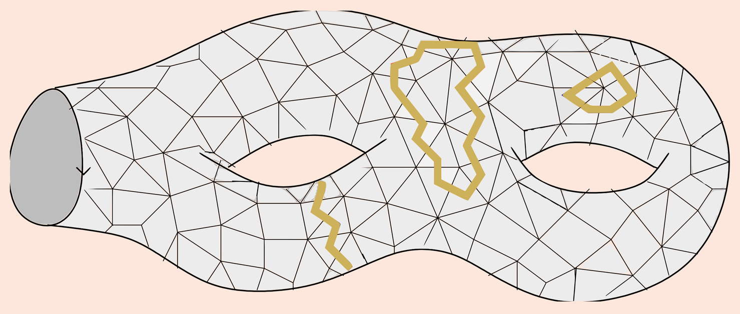
1 - $\mathcal{O}(\mathfrak{n})$ model

$$\mathrm{W}_k^{(g)}(x_1,\ldots,x_k) = \sum \frac{1}{\text{number of automorphisms}} \frac{t^v\, t_3^{n_3}\cdots t_d^{n_d}\, \mathfrak{n}^L\, \mathfrak{z}^\ell}{x_1^{j_1+1}\cdots x_k^{j_k+1}}$$

$\sum$ ranges over connected genus g discrete surface with k boundaries, built with:

- v vertices
- n_j j-gons ($j \geq 3 \geq d$, d fixed but arbitrary)
- ℓ triangles carrying a piece of path forming exactly L loops
- a marked j_i-gon ($j_i \geq 1$) with a marked edge as i-th boundary ($1 \leq i \leq k$)



For $\mathfrak{n} \in \mathbb{N}$, it admits a representation as a formal hermitian matrix model [1, 6]:

$$\mathrm{d}\lambda(M, A_1, \ldots, A_\mathfrak{n}) = \overbrace{\mathrm{d}M\, \mathrm{d}A_1 \cdots \mathrm{d}A_\mathfrak{n}}^{\text{Lebesgue measure on } \mathcal{H}_N^{1+\mathfrak{n}}} \quad \exp\left[-\frac{N}{t}\mathrm{Tr}\left(\mathrm{V}(M) + \sum_{i=1}^{\mathfrak{n}}\left(\frac{1}{2\mathfrak{z}} + M\right)A_i^2\right)\right]$$

$$\left\langle \prod_{x_i \in K} \mathrm{Tr}\frac{1}{x_i - M}\right\rangle_c = \sum_{g=0}^{\infty}\left(\frac{N}{t}\right)^{2-2g-k} \mathrm{W}_k^{(g)}(K)$$

- $\mathrm{V}(x) = \frac{x^2}{2} - \sum_{j=3}^{d} \frac{t_j}{j} x^j$
- equalities between formal series in t, of polynomials in t_j's and rational functions in x_i's
- c for cumulant
- analytic continuation for $\mathfrak{n} \in \mathbb{R}$

We proved an algorithm to compute all $\mathrm{W}_k^{(g)}$'s

2 - Interest for the $\mathcal{O}(\mathfrak{n})$ model

Critical points ▷ Exhibits critical points [2] different from pure gravity at $t_c > 0$, for $0 < |n| \leq 2$
For $\mathfrak{n} = -2\cos\pi\mathfrak{g}$ ($\mathfrak{g} \in]0,1[$), several continuum limits

▷ Believed to be CFT$_\mathfrak{c} \otimes$ gravity, with
$\mathfrak{c} = 1 - 6\left(\tilde{\mathfrak{g}}^{1/2} - \tilde{\mathfrak{g}}^{-1/2}\right)^2$ where $\tilde{\mathfrak{g}} = \varepsilon(1-\mathfrak{g}) + 2p + 1$

→ reach non rational CFT's by the continuum limit of a microscopic model

Combinatorics → Counting discrete surfaces with additional structure

▷ Duality to $\mathfrak{q} = \mathfrak{n}^2$ Potts model
▷ Fully packed loops ↔ dimer configurations when $\mathrm{V}(x) = x^2/2$

Matrix models → A direction of generalization of the algebraic geometry tools developed for the 1-matrix model ($\mathfrak{n} = 0$) [5]

3 - The method of loop equations

Loop equations = change of variables in the matrix integrals
Powerful way to prove automatically combinatorial recursion relations [7]

Lemma 0 ▷ When $x \to \infty$
$$\begin{cases} \mathrm{W}_1^{(0)}(x) \sim t/x \\ \mathrm{W}_k^{(g)}(x, J) \in O(1/x^2) \text{ else} \end{cases}$$

Combinatorial lemma ▷ in each variable (for $k \geq 1$), $\mathrm{W}_k^{(g)}(x_1,\ldots,x_k)$ is holomorphic with one cut $[a(t), b(t)] \subseteq \mathbb{C}$
▷ When $x \to a_i \in \{a(t), b(t)\}$ $\quad \mathrm{W}_1^{(0)}(x) - \mathrm{W}_1^{(0)}(a_i) \propto \sqrt{(x - a_i)}$

There exists a set of loop equations determining uniquely $\mathrm{W}_k^{(g)}$ satisfying these analytical properties

4 - Analytical properties of $\mathrm{W}_k^{(g)}$

The one-cut property implies $\forall x \in [a(t), b(t)]$ and $\epsilon \to 0$:

[3] $\mathrm{W}_1^{(0)}(x + i\epsilon) + \mathrm{W}_1^{(0)}(x - i\epsilon) + \mathfrak{n}\,\mathrm{W}_1^{(0)}(-x) = \mathrm{V}'(x)$

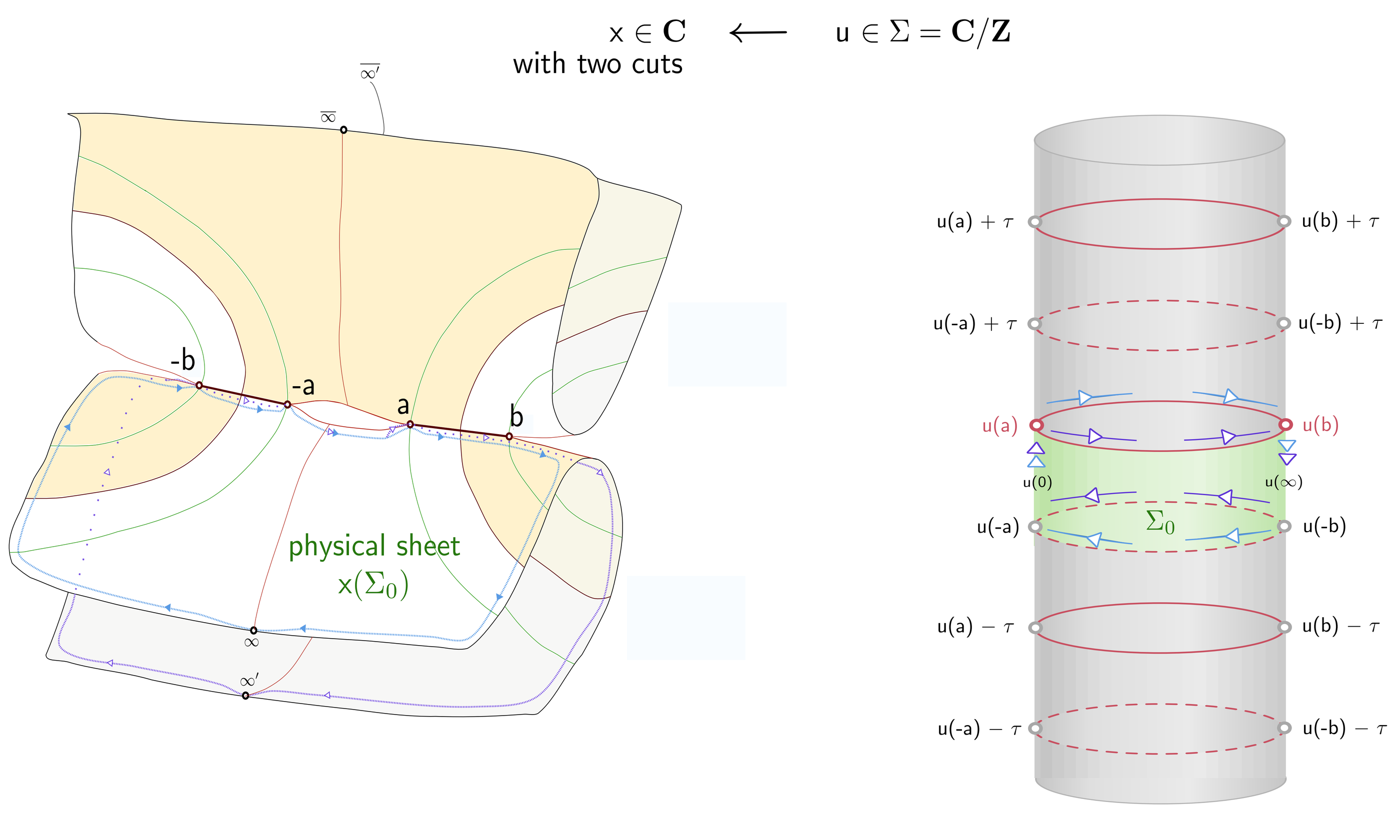
[3] $\mathrm{W}_2^{(0)}(x_1 + i\epsilon, x_2) + \mathrm{W}_2^{(0)}(x_1 - i\epsilon, x_2) + \mathfrak{n}\,\mathrm{W}_2^{(0)}(-x_1, x_2) = \frac{-1}{(x_1 - x_2)^2}$

[4] $\mathrm{W}_k^{(g)}(x_1 + i\epsilon, J) + \mathrm{W}_k^{(g)}(x_1 - i\epsilon, J) + \mathfrak{n}\,\mathrm{W}_k^{(g)}(-x_1, J) = 0$ else

→ Riemann-Hilbert problem: solution in terms of functions on a spectral curve $\mathcal{S}$

Def. of spectral curve data $\mathcal{S} = (\Sigma, x, y)$, where

- Σ is a Riemann surface
- x and y are two meromorphic functions on Σ



5 - Topological recursion

- In [4], we have extended the topological recursion of [5] to the $\mathcal{O}(\mathfrak{n})$ model
- One recovers the formalism valid for the 1-hermitian matrix for $\mathfrak{n} = 0$
- Once the spectral curve is found, few modifications arise when $\mathfrak{n} \neq 0$
  Here, one can construct $x : \mathbb{C} \to \mathbb{C} \cup \{\infty\} \setminus \{\text{cuts}\}$, such that $\lim_{\epsilon\to 0}\left(\mathrm{W}_1^{(0)}(x(u) + i\epsilon) - \mathrm{W}_1^{(0)}(x(u) - i\epsilon)\right)$
  defined on [u(a),u(b)], can be analytically continued as a monovalued function on Σ → this function is called $y(u)$
- $\mathrm{W}_k^{(g)}(x(u_1), \ldots, x(u_k))\mathrm{d}x(u_1)\cdots\mathrm{d}x(u_k)$ defines by analytic continuation meromorphic forms on Σ : $\quad \omega_k^{(g)}(u_1, \ldots, u_k)$

Physical sheet We choose a maximal open set $\Sigma_0 \subseteq \Sigma$ such that x maps bijectively Σ_0 to $\mathbb{C} \cup \{\infty\} \setminus \{\text{cuts}\}$

Branch points We choose a set of simple zeroes of dx: $\{a_i\} \subseteq \overline{\Sigma}_0$

Local involution We define $\overline{u}$ defined locally around a_i by $x(\overline{u}) = x(u)$

Bergman kernel Unique differential two form on Σ
with prescribed double poles without residues at $x(u_1) = \pm x(u_2)$
More precisely, such that, $\forall u_1, u_2 \in \Sigma_0$:

$$\mathcal{B}(u_1, u_2) = \omega_2^{(0)}(u_1, u_2) + \frac{\mathrm{d}x(u_1)\mathrm{d}x(u_2)}{4 - \mathfrak{n}^2}\left(\frac{2}{(x(u_1) - x(u_2))^2} - \frac{\mathfrak{n}}{(x(u_1) + x(u_2))^2}\right)$$

Recursion kernel We define globally in u_1, locally around a_i in u:

$$\mathcal{K}(u_1, u) = -\frac{1}{2}\frac{\int_{\overline{u}}^{u}\mathcal{B}(u_1, u')}{(y(u) - y(\overline{u}))\mathrm{d}x(u)}$$

Correlation forms For $2g - 2 + k > 0$, $\omega_k^{(g)}$ can be reached with a string of $2g - 2 + k$ residues at branch points.

$$\omega_k^{(g)}(u_1, J) = \sum_i \mathop{Res}_{u\to a_i} \mathcal{K}(u_1, u)\left[\text{"}\omega_{k+1}^{(g-1)}(u, \overline{u}, J)\text{"} + \sum_{I \subseteq J,\, h}^{\prime} \tilde{\omega}_{|I|+1}^{(g)}(u, I)\,\tilde{\omega}_{k-|I|}^{(g-h)}(\overline{u}, J \setminus I)\right]$$

Free energies For $g \geq 2$, $\mathrm{F}_g = \mathrm{W}_0^{(g)}$ can be reached with a string of $2g - 2$ residues.
Let ϕ_i be a local primitive of $y\mathrm{d}x$ around a_i:

$$\mathrm{F}_g = \frac{1}{2 - 2g}\sum_i \mathop{\mathrm{Res}}_{u\to a_i}\left(\phi_i(u)\,\omega_1^{(g)}(u)\right)$$

- We also have $\mathfrak{n}$-deformed expressions for F_0 and F_1 (these expressions are always more involved)
- Properties wrt infinitesimal deformation [5] of $\mathcal{S}$ are preserved

The topological recursion commutes with (singular) limits of spectral curves:

$$\mathcal{S}_\alpha : \begin{cases} x(z) = c + \alpha\, x^*(z) \\ y(z) = \alpha^\mu\, y^*(z) \end{cases} \quad \Longrightarrow \quad \begin{array}{l} \mathrm{W}_k^{(g)}[\mathcal{S}_\alpha](c + \alpha\, x_1^*, \ldots, c + \alpha\, x_k^*) \sim \\ \alpha^{(2-2g-k)(\mu+1)-k}\, \mathrm{W}_k^{(g)}[\mathcal{S}^*](x_1^*, \ldots, x_k^*) \end{array}$$

when $\alpha \to 0$ for $2g-2+k \geq 0$

6 - Study of critical points

- The most important one is the *string susceptibility*: $\partial_t^2 \mathrm{F}_0 \propto (1 - t/t_c)^{-\gamma_{\mathrm{str}}}$
- Planar ($g = 0$) critical exponents and amplitudes were already known for small k
- KPZ scaling expected from Liouville th. predicts a $(2 - 2g - k)$ dependance of the general exponents

Pure gravity = No macroscopic loops in the continuum limit already exists in the 1-hermitian matrix model
Reached by *blowing up* $\mathcal{S}$ around a branch pt which becomes a zero of order p of dx at $t = t_c$
$\mu = p + \frac{1}{2}$, $\quad \gamma_{\mathrm{str}} = -1/(p+1)$

Dense phase (critical pt) = Macroscopic loops filling densely the surface
Reached when $a \to 0$ (defines $t \to t_c$), by blowing up $x \sim a\,x^*$
$\mu = \mathfrak{g}$, $\quad \gamma_{\mathrm{str}} = 1 - \frac{1}{\mathfrak{g}}$

Dilute phases (multicritical pt) = Macroscopic loops and regions dominated by gravity simultaneously
Reached when V is properly tuned while $a \to 0$, by blowing up $x \sim a\,x^*$
Parametrization by $(\varepsilon, p) \in \{\pm\} \times \mathbb{N}$ with $(\varepsilon, p) \neq (-1, 0)$
$\mu = \varepsilon(1 - \mathfrak{g}) + 2p + 1$, $\quad \gamma_{\mathrm{str}} = -\frac{2(1-\mathfrak{g})}{\mathfrak{g} + \varepsilon(1-\mathfrak{g}) + 2p + 1}$

Critical spectral curve
$$\begin{cases} x^*(z) = \mathrm{ch}(z) \\ y^*(z) = \sum_{p'} A_{p'}\,\mathrm{sh}[(\mathfrak{g} + 2p' + 1)z + 2i\pi\mathfrak{g}] \end{cases}$$

- $\mathrm{W}_k^{(g)}[\mathcal{S}^*]$ are the bulk correlators in CFT$_\mathfrak{c} \otimes$ gravity (amplitudes for FZZT branes)
- Proves KPZ scaling with $a^2 \to 0$ interpreted as a cosmological constant

7 - Conclusion

- A microscopic model which has a non algebraic spectral curve is shown to satisfy the topological recursion
- Perspective : computing all possible boundary conditions (insertion of paths on boundaries, e.g. [8, 9, 10]) as done in the 2-hermitian matrix model
- Generalization to β-ensembles coupled to an $\mathcal{O}(\mathfrak{n})$ model ?

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