Extreme Value Statistics of 1/f Noises generated by 2d Gaussian Free Field:

Statistical Mechanics Approach

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References:

Y V F, J-P Bouchaud : J. Phys.A: Math.Theor 41 (2008), 372001 (12pp)

Y V F, P Le Doussal , and A Rosso : J. Stat. Mech. (2009) P10005 (31pp)

Summary of standard extreme-value statistics:

• Let $z_1, \dots, z_N$ be i.i.d. random variables with probability density function $p(z)$. Let $y_N = max_{i=1,\dots,N}\{z_n\}$ be **maximum** of the set, and $F_N(y) = Prob(y_N < y)$ be the **distribution of the maximum**. Then for $N \gg 1$ the distribution approaches a **scaling form** $F_N(y) \approx F_\infty\left[(y + a_N)/b_N\right]$ where a_N, b_N depend on $p(z)$ but the shape of F_∞ is **universal**, and given by

$$F_\infty(y) = \begin{cases} e^{-e^{-y}}, \quad \forall y & \textbf{Gumbel class: } z < \infty \text{ and } p(z \to \infty) \sim Ae^{-z^\alpha}, \ \alpha > 0 \\ e^{-y^{-\alpha}}, \ y \ge 0 & \textbf{Fréchet class: } z < \infty \text{ and } p(z \to \infty) \sim Az^{-(\alpha+1)} \\ e^{-[-y]^\alpha}, \ y \le 0 & \textbf{Weibull class: } z < a \text{ and } p(z \to a) \sim A(a - z)^{(\alpha-1)} \end{cases}$$

The result is rather robust if variables are short-range correlated. In particular, for Gaussian-distributed variables with $< z_i >= 0$ the **Gumbel** distribution is known to be valid as long as $C(|i-j|) = \langle z_i z_j \rangle \lesssim const/\ln|i-j|$ for $|i-j| \gg 1$.

Very few explicit results exist for extrema of **strongly correlated variables**, as e.g. for Brownian motion by **Majumdar & Comtet**, or for the largest eigenvalues of random matrices by **Tracy & Widom**.

Gaussian Free Field: definition:

• Given any domain $\mathbf{D}$ consider the **Laplace** operator $-\Delta$ and denote $\mathbf{e}_j(\mathbf{x})$ and $\lambda_j > 0$ for $j = 1, 2, \ldots, \infty$ its eigenfunctions/eigenvalues corresponding to the Dirichlet boundary conditions. Then the functions $\tilde{\mathbf{e}}_j(\mathbf{x}) = \frac{1}{\sqrt{\lambda_j}}\mathbf{e}_j(\mathbf{x})$ form an orthonormal basis of the Hilbert space $\mathtt{H}$ w.r.t. the so-called **Dirichlet** scalar product

$$(f,g) = \int_{\mathbf{D}} (\nabla f \cdot \nabla g)\, d^N\mathbf{x} = -\int_{\mathbf{D}} (f \cdot \Delta g)\, d^N\mathbf{x}$$

for functions $f(\mathbf{x})$ on $\mathbf{D}$ vanishing at the boundary $\partial\mathbf{D}$. Introduce now a set $\zeta_j,\; j = 1, 2, \ldots, \infty$ of standard Gaussian i.i.d. real variables with mean zero and unit variance each: $\langle\zeta_j\rangle = 0$, $\langle\zeta_j^2\rangle = 1$. Then the **GFF** $V(\mathbf{x})$ on the domain $\mathbf{x} \in \mathbf{D}$ is defined as the formal sum $V(\mathbf{x}) = \sum_{j=1}^{\infty} \zeta_j\, \tilde{\mathbf{e}}_j(\mathbf{x})$. We immediately see that it is a **Gaussian field** with the two-point correlation function (**covariance**) given by the **Green function** $G(\mathbf{x}_1, \mathbf{x}_2) = -\left(\Delta^{-1}\right)(\mathbf{x}_1, \mathbf{x}_2)$ of the Laplace operator on $\mathbf{D}$:

$$\langle V(\mathbf{x}_1)V(\mathbf{x}_2)\rangle = \sum_{j=1}^{\infty} \frac{1}{\lambda_j}\mathbf{e}_j(\mathbf{x}_1)\mathbf{e}_j(\mathbf{x}_2) = G(\mathbf{x}_1, \mathbf{x}_2)$$

Gaussian Free Field: examples:

• **GFF** on the interval $\mathbf{D} = [0,1]$. Eigenf./eigenv. for the Laplacian $\Delta = -\frac{d^2}{dx^2}$ (Dirichlet b.c.): $e_n(x) = \sqrt{2}\sin n\pi x, \lambda_n = \pi^2 n^2$. The corresponding GFF is given by the random Fourier series $V(x) = \sum_{n=1}^{\infty} \zeta_n \frac{\sqrt{2}}{\pi n} \sin n\pi x$, with the covariance given by the Green function $G(x_1,x_2) = \min(x_1,x_2)[1-\max(x_1,x_2)]$ - **Brownian bridge**.

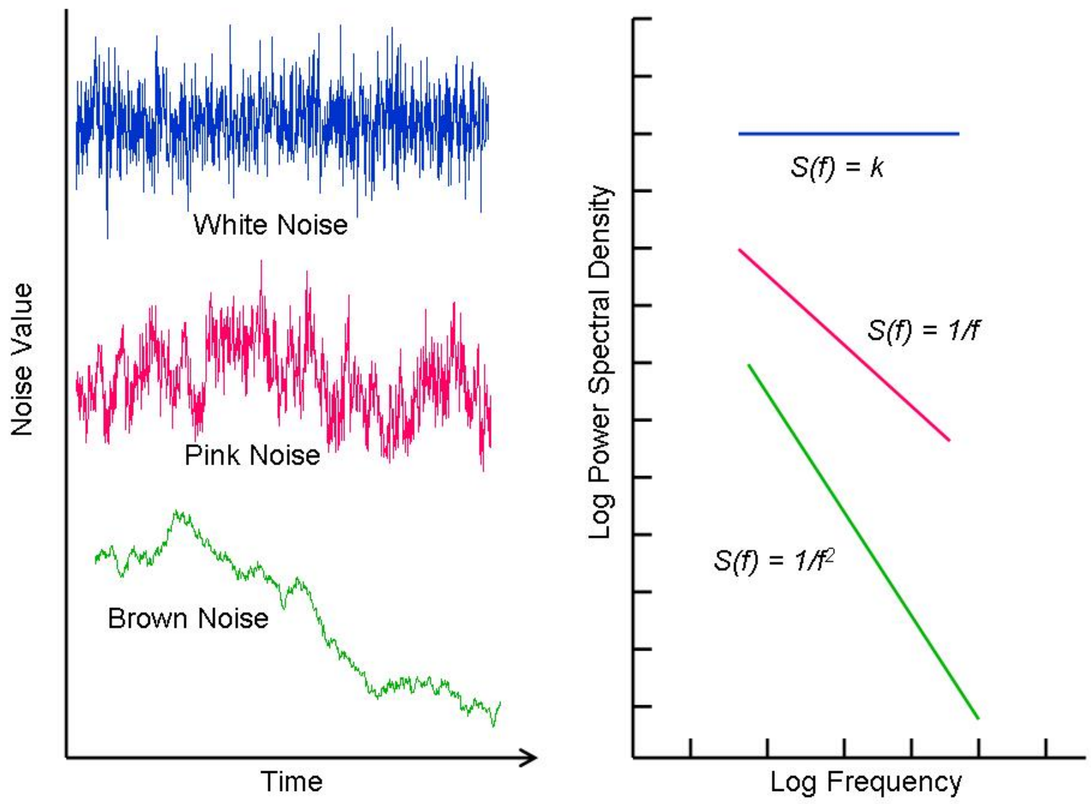
• **GFF** on the two-dimensional disk: $\mathbf{D} = |z| < L$ where $z = x + iy$. The Green function is given by $G(z_1,z_2) = -\frac{1}{2\pi}\ln\frac{L|z_1-z_2|}{L^2 - z_1 z_2}$. In particular, for any two points $|z_{1,2}| \ll L$ (i.e. well inside the disk) we recover the **full-plane** formula

$$G(z_1,z_2) = -\frac{1}{2\pi}\ln\frac{|z_1-z_2|}{L} \Leftrightarrow \mathcal{P}\left[V(\mathbf{x})\right] \propto \exp -\frac{1}{2}\int \left[\nabla V(\mathbf{x})\right]^2 d^2\mathbf{x}$$

• Using the full-plane logarithmic **GFF** we can construct various one-dimensional Gaussian random processes with **logarithmic** correlations. In particular, sampling the values of such **GFF** along a circle of unit radius with coordinates $z = e^{it}$, $t \in [0, 2\pi)$ we get a Gaussian process with mean zero and the covariance

$$\langle V(t_1)V(t_2)\rangle = -\frac{1}{2\pi}\ln|e^{it_1} - e^{it_2}|$$

Such a process turns out to be equivalent to the random Fourier series of the form $V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[v_n e^{int} + \overline{v}_n e^{-int} \right]$, where $v_n, \overline{v}_n$ are i.i.d. complex Gaussian variables with variance $\langle v_n \overline{v}_n \rangle = 1$. As the power associated with a given Fourier harmonic with index n decays like $1/n$ such signals are known as $1/f$ noises believed to be ubiquitous in Nature.



Problem:

- Given an instance of the full-plane $2D$ **Gaussian free field**:

$$\mathcal{P}\left[V(\mathbf{x})\right] \propto \exp -\frac{1}{8\pi g^2}\int \left[\nabla V(\mathbf{x})\right]^2 \, d^2\mathbf{x}$$

characterized by the covariance

$$\langle V(\mathbf{x_1})V(\mathbf{x_2})\rangle = -2g^2 \ln|\mathbf{x_1}-\mathbf{x_2}|$$

we wish to understand the statistics of its **minima/maxima** along various curves in the plane, and ultimately in various planar domains.

- The problem turns out to be intimately connected to the mechanism of **freezing transitions** in disordered systems theory (Random Energy Models, Dirac fermions in random magnetic field). It has also interesting relations to **Liouville Quantum Gravity** $\&$ conformal field theory, to **multifractal** random measures, $1/f$ **noises**, and processes arising in turbulence and mathematical finance, as well as to various aspects of **Random Matrix Theory**.

Idea of the method: We concentrate on considering samples of the full-plane Gaussian Free Field (2d**GFF**) along **planar curves** $\mathcal{C}$ parametrised by $\mathbf{x}(t) = (x(t), y(t))$ with real $t \in [a, b]$.
Given a measure $d\mu_\rho(t) = \rho(t)\, dt$, we consider the integral

$$Z_\beta = \epsilon^{\beta^2 g^2} \int_a^b e^{-\beta V_\epsilon(\mathbf{x}(t))} d\mu_\rho(t), \quad \beta > 0$$

where $V_\epsilon(\mathbf{x})$ is the regularized version of the 2d**GFF** with a short scale cutoff $\epsilon \ll 1$, i.e. zero mean and the covariance

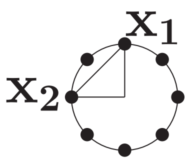
$$\langle V_\epsilon(\mathbf{x}) V_\epsilon(\mathbf{x}') \rangle = -2g^2 \ln |\mathbf{x} - \mathbf{x}'|_\epsilon = \begin{cases} -2g^2 \ln |\mathbf{x} - \mathbf{x}'|, & |\mathbf{x} - \mathbf{x}'| > \epsilon \\ 2g^2 \ln(1/\epsilon), & |\mathbf{x} - \mathbf{x}'| < \epsilon \end{cases}$$

The integral is to be interpreted as the **partition function** of the associated **Random Energy Model** at the temperature $T = \beta^{-1}$. This is to be studied in the limit $\epsilon \to 0$.

Guiding example: CIRCULAR LOGARITHMIC MODEL:

Let the contour $\mathcal{C}$ be the unit circle: $x(t) = \cos t, y(t) = \sin t$, with $t \in [0, 2\pi)$. Sample the 2d Gaussian Free Field at M equidistant points along the circle with $t_k = \frac{2\pi}{M}(k-1),\ k = 1, \ldots, M$

As the distance $|\mathbf{x}_1 - \mathbf{x}_2|$ between a pair of points is simply $2|\sin\frac{t_1 - t_2}{2}|$, we deal with the collection of M normally distributed variables with covariances



$$\langle V_k V_m \rangle = -2g^2 \ln |2 \sin \frac{2\pi}{M}(k-m)|, \quad \text{for} \quad k \neq m$$

We have to choose the variance accordingly:

$$\langle V_k^2 \rangle = 2g^2 \ln M + W, \quad \text{with any} \quad W > 0$$

Equivalently, we consider $2\pi-$ periodic Gaussian $1/f$ **noise**:

$$V(t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} (v_n e^{int} + \overline{v}_n e^{-int}) \quad \text{with i.i.d. coefficients } \langle v_n \overline{v}_k \rangle = g^2 \delta_{n,k}$$

Observation: The positive integer moments $\langle Z^n(\beta)\rangle$, $n = 1, 2, \ldots$ of the partition function $Z(\beta) = \sum_{i=1}^{M} e^{-\beta V_i}$ for the circular logarithmic model in the high-temperature phase $\gamma = \beta^2 g^2 < 1$ turn out to be given in the thermodynamic limit $M \gg 1$ by

$$\langle Z^n_{circ}(\beta)\rangle = \begin{cases} M^{1+\gamma n^2} O(1) & n > 1/\gamma \\ M^{(1+\gamma)n} D_n(\gamma) & n < 1/\gamma \end{cases}$$

where $D_n(\gamma)$ is the **Dyson-Morris** Integral

$$D_n(\gamma) = \frac{1}{(2\pi)^n} \int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_n \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^{-2\gamma} = \frac{\Gamma(1 - n\gamma)}{\Gamma^n(1 - \gamma)}$$

Aim: to reconstruct the distribution of the partition function $P(Z)$ from its moments in the high temperature phase $\gamma \leq 1$.

Outcome of the analysis:

The probability density $\mathcal{P}(Z)$ of the partition function $Z_{circ}(\beta) \equiv Z$ in the high-temperature phase $\gamma = \beta^2 g^2 < 1$ consists of two pieces. The **"body"** of the distribution is given by:

$$\mathcal{P}(Z) = \frac{1}{\gamma}\frac{1}{Z}\left(\frac{Z_e}{Z}\right)^{\frac{1}{\gamma}} e^{-\left(\frac{Z_e}{Z}\right)^{\frac{1}{\gamma}}}, \quad Z \ll M^2$$

which has a pronounced maximum at $Z \sim Z_e = \frac{M^{1+\gamma}}{\Gamma(1-\gamma)} \ll M^2$, and the powerlaw decay at $Z_e \ll Z \ll M^2$.

At $Z \gg M^2$ the above expression is replaced by the **lognormal tail**:

$$\mathcal{P}(Z) = \frac{M}{\sqrt{4\pi\gamma \ln M}}\frac{1}{Z} f\left(\frac{1}{2}\frac{\ln Z}{\ln M}\right) e^{-\frac{1}{4\ln M\gamma}\ln^2 Z} \quad \text{where } f(x) \sim O(1) \text{ for } x \sim O(1)$$

Now we define $z = Z/Z_e$, put the coupling constant $g = 1$ and consider the generating function

$$g_\beta(x) = \left\langle \exp(-e^{\beta x} z) \right\rangle_{M \gg 1}, \quad \beta = 1/T$$

Freezing scenario: In the high-temperature phase $\beta < \beta_c = 1$ the generating function $g_\beta(x)$ can be found explicitly and turned out to satisfy a remarkable **duality relation**:

$$g_\beta(x) = \int_0^\infty dt \exp\left\{-t - e^{\beta x} t^{-\beta^2}\right\}, \Rightarrow g_\beta(x) = g_{\frac{1}{\beta}}(x) .$$

This however does not allow to continue to $\beta > \beta_c$ regime. The phase transition at $\beta = \beta_c$ is believed to be described by the following **freezing scenario**: $g_\beta(x)$ **freezes** to the **temperature independent** profile $g_{\beta_c}(x)$ in the "glassy" phase $T \leq T_c$. The scenario is supported by

(i) a heuristic **real-space renormalization group arguments** for the logarithmic models (**Carpentier, Le Doussal '01**) revealing an analogy to the **travelling wave** analysis of polymers on disordered trees (**Derrida, Spohn 1989**)

(ii) duality which implies

$$\partial_\beta g_\beta(x)|_{\beta=\beta_c^-} = 0 \quad , \text{ for all } x$$

showing that the "temperature flow" of this function vanishes at the critical point $\beta = \beta_c = 1$

(iii) our **numerics**.

Assuming validity of such scenario for the problem in hand, one finds the frozen profile for the circular model:

$$g_{\beta_c}^{circ}(x) = 2e^{x/2}K_1(2e^{x/2})$$

where $K_1(z)$ is the Macdonald function. This allows to reconstruct the **distribution of the free energy** $f = -\beta^{-1}\ln z$ for any $T < T_c$. The corresponding formula takes a form of an infinite series:

$$\mathcal{P}_{\beta>\beta_c}^{CLM}(f) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isf}\,\frac{1}{\Gamma(1+\frac{is}{\beta})}\Gamma^2\left(1+\frac{is}{\beta_c}\right)\,ds$$

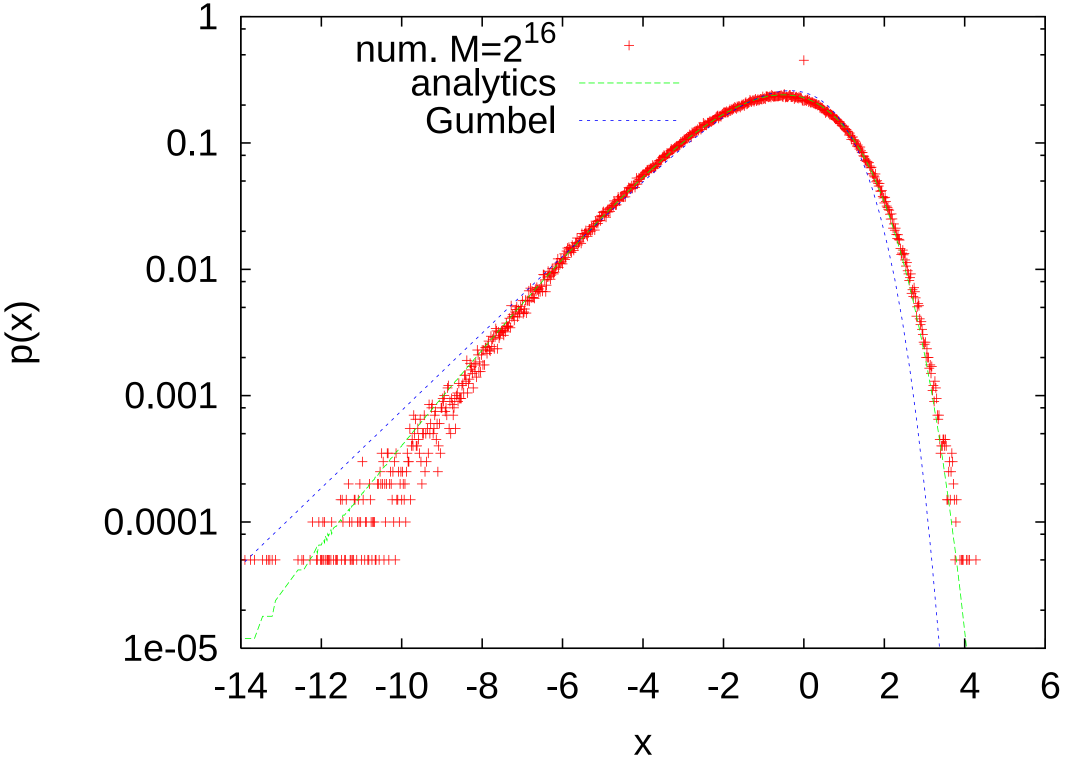
$$= -\frac{d}{df}\left[1+\sum_{n=1}^{\infty}\frac{e^{n\beta_c f}}{n!(n-1)!\Gamma\left(1-n\frac{\beta_c}{\beta}\right)}\left(\beta_c f+\frac{1}{n}-2\psi(n+1)+\frac{\beta_c}{\beta}\psi\left(1-n\frac{\beta_c}{\beta}\right)\right)\right]$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$. In the zero temperature limit $\beta \to \infty$ the free energy distribution yields the **extreme value probability density.**

The minimum of the random potential is simply given by $V_{min} = -\lim_{T\to 0} f = const + x$, with known $const$ and the probability density of x related to the frozen profile $g_{\beta_c}(x)$ by

$$p(x) = -g'_{\beta_c}(x) = -\frac{d}{dx}\left[2e^{x/2}K_1(2e^{x/2})\right] \quad (1)$$

This is different from **Gumbel** distribution $p_{Gum}(x) = -\frac{d}{dx}\left[\exp -Be^{Ax}\right]$.



Distribution of extremes: we compare three distributions: (i) the histrogram for ensemble of 10^6 realizations of the Gaussian free field sampled at $M = 2^{16}$ points equispaced along the unit circle, (ii) the analytical prediction (1), and (iii) the Gumbel distribution for the mean & variance given by (1)

From circles to intervals:

For integers $n = 1, 2, ..,$ a well defined and universal $\epsilon \to 0$ limits exist for the moments of the partition function

$$\left\langle Z^n_{[0,1]} \right\rangle = \int_0^1 \ldots \int_0^1 \prod_{1 \le i < j \le n} |x_i - x_j|_\epsilon^{-2\gamma} \prod_{i=1}^{n} x_i^a (1 - x_i)^b \, dx_i,$$

as long as $\gamma = \beta^2 < 1/n$, in which case they are given by celebrated **Selberg integral** formula. Defining $z = \Gamma(1 - \gamma) Z$ we obtain the moments

$$z_n = \left\langle z^n_{[0,1]} \right\rangle = \prod_{j=1}^{j=n} \frac{\Gamma[1 + a - (j - 1)\gamma]\Gamma[1 + b - (j - 1)\gamma]\Gamma(1 - j\gamma)}{\Gamma[2 + a + b - (n + j - 2)\gamma]}$$

To restore the corresponding **probability density** we need to know the generic moments $M_\beta(s) = \left\langle z^{1-s} \right\rangle, \quad M_\beta(1) = 1$ for any **complex** s, at fixed inverse temperature β and parameters a, b. Given those moments, we can exploit the relation between the moments $M_\beta(s)$ and the generating function $g_\beta(x)$

$$\ln \left[- \int_{-\infty}^{\infty} g'_\beta(x) \, e^{xs} \, dx \right] = \ln M_\beta(1 + \frac{s}{\beta}) + \ln \Gamma(1 + \frac{s}{\beta})$$

High temperature phase for $[0,1]$ interval without weight: $a=b=0$:

Let $\beta<\beta_c=1$. Introduce the parameter $Q=\beta+1/\beta$ and define the function $G_\beta(x)$ for $\Re(x)>0$ by :

$$\ln G_\beta(x)=\frac{x-Q/2}{2}\ln(2\pi)+\int_0^\infty\frac{dt}{t}\left(\frac{e^{-\frac{Q}{2}t}-e^{-xt}}{(1-e^{-\beta t})(1-e^{-t/\beta})}+\frac{e^{-t}}{2}(Q/2-x)^2+\frac{Q/2-x}{t}\right)$$

This function is **self-dual:** $G_\beta(x)=G_{1/\beta}(x)$ and satisfies the functional relation

$$G_\beta(x+\beta)=\beta^{1/2-\beta x}(2\pi)^{\frac{\beta-1}{2}}\Gamma(\beta x)\,G_\beta(x)$$

see e.g. [**Fateev, Zamolodchikov, Zamolodchikov 2000**]. Note that $G_{\beta=\beta_c=1}(s)=G(s)$ is the well-known **Barnes function** satisfying $G(s+1)=\Gamma(s)G(s),\ G(1)=1$. The function $G_\beta(s)$ can be exploited to find the required analytical continuation of moments $M_\beta(s)=\left\langle z^{1-s}\right\rangle$ to complex values of s for any temperature above (and at) critical.

In this way we arrive to

$$M_\beta(s) = A_\beta 2^{(s-1)(2+\beta^2(2s+1))}\pi^{1-s}$$
$$\times \frac{\Gamma(1+\beta^2(s-1))G_\beta(\frac{\beta}{2}+\frac{1}{\beta}+\beta s)G_\beta(\frac{3}{2\beta}+\beta s)G_\beta(\frac{\beta}{2}+\frac{3}{2\beta}+\beta s)}{G_\beta(\beta+\frac{2}{\beta}+\beta s)G_\beta(\frac{1}{\beta}+\beta s)^2}$$

with $A_\beta = \frac{G_\beta(\frac{1}{\beta}+\beta)^2 G(2\beta+\frac{2}{\beta})}{G_\beta(\frac{3\beta}{2}+\frac{1}{\beta})G_\beta(\frac{3}{2\beta}+\beta)G_\beta(\frac{3\beta}{2}+\frac{3}{2\beta}))}$. To guarantee that we have found the correct continuation, we have checked

(i) positivity: $M(s)$ given above is finite and positive on the interval $s \in [0, +\infty[$ that is all real moments $n = 1 - s < 1$ exist.

(ii) convexity: on this interval $\partial_s^2 \ln M(s) > 0$.

(iii) For **integer** values of s gives back known positive/negative moments.

We can use the above expression to extend the **duality relation**:

$$g_\beta(x) = g_{\frac{1}{\beta}}(x)\,.$$

to the case of the interval $[0, 1]$.

Under the **freezing hypothesis** we extract the frozen profile $g_{\beta_c}(x)$. For the general case the expression can be obtained as expansion in powers of e^x for $x \to -\infty$. For example, for $a = b = 0$

$$g_{\beta_c}(x \to -\infty) = 1 + (x + A')e^x + (A + By + Cx^2 + \frac{1}{6}x^3)e^{2x} + \ldots \quad (2)$$

with $A' = 2\gamma_E + \ln(2\pi) - 1$ and $C = -0.253846$, $B = 1.25388$, $A = -5.09728$. For the special case $a = b = -1/2$ we obtain the closed form expression:

$$g_{\beta_c}(x) = \frac{\pi}{4} \int_{-\infty}^{+\infty} \frac{dt}{\sqrt{2\pi}} e^{-\frac{t^2}{2} - 2\sqrt{\ln 2}t} \int_{e^x}^{\infty} \left(1 - \frac{e^x}{u}\right) e^{-\sqrt{\pi u/2}\, e^{-\sqrt{\ln 2}t}} du$$

Although these expressions are different from the circle case, the universal **Carpentier-Le Doussal tail** for the probability density of **extreme values**

$$p(x \to -\infty) = -g'_{\beta_c}(x \to -\infty) \sim -xe^x$$

is shared by all these distributions. It has its origin in the characteristic tail of the partition function density $P(z \gg 1) \propto 1/z^2$ developed at criticality, with the first moment $< z >$ becoming **infinite**.

Conclusions $\&$ Discussions:

- Using the methods of statistical mechanics we were able to extract the explicit expressions for distributions of extrema of the Gaussian Free Field sampled along (i) circles of unit radius and (ii) intervals of unit length. The distributions are manifestly non-Gumbel and show universal backward tail. The results are expected to describe extreme value statistics for $1/f$ signals, and in this way could be relevant for spectral fluctuations of quantum chaotic systems and Riemann zeta-function (cf. e.g. Relaño *et al* PRL 89 (2002) 244102).
- We revealed a "duality relation" satisfied by specific generating function of scaled free energies everywhere in the high temperature phase. The same object is expected to show freezing of its shape at the critical temperature. It is tempting to conjecture relation between freezing and self-duality.

Our method was based on a few assumptions, most importantly **(i)** freezing scenario for REM-type models, and **(ii)** ability to continue Selberg integrals away from positive integers to the complex plane (can be put on the rigorous basis by a method developed recently in **D. Ostrovsky Comm. Math. Phys. 288 (2009) 287-310**)

It remains a challenge:

- to verify/justify the freezing scenario
- to understand universality of the results for other $1d$ curves
- to access extreme value statistics of GFF in 2D domains.

Related work in progress:

Statistics of velocities in **decaying Burgers turbulence** with correlated initial conditions $< v(x)v(x') >\sim |x - x'|^{-2}$.