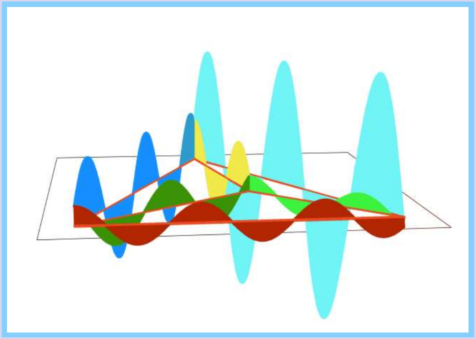
The nonlinear Schrödinger equation (NLSE) on metric graphs

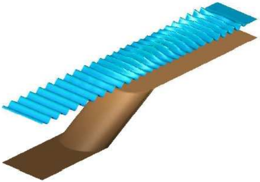


Sven Gnutzmann (Nottingham) & Uzy Smilansky (Weizmann)

Workshop on Random-Matrix Theory, Brunel, 19 December 2009

Physical applications of the NLSE

- Shallow water waves



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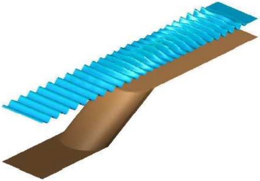
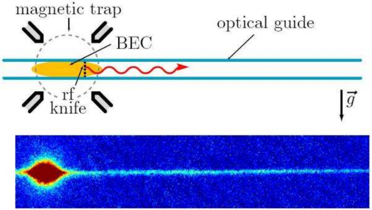


- Bose-Einstein Condensates and Superfluids



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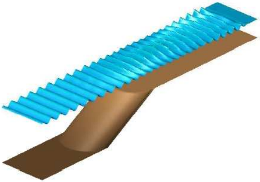

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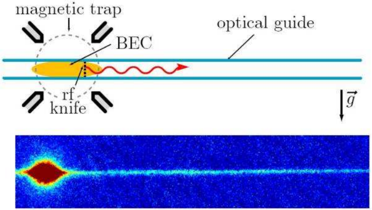
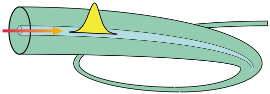
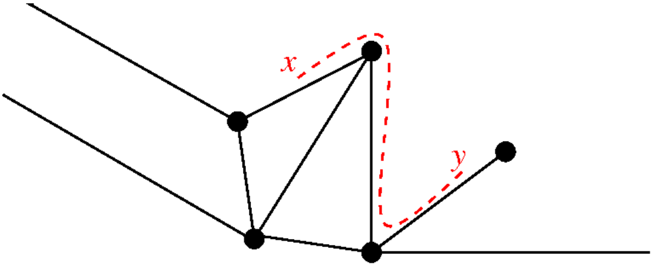


- 

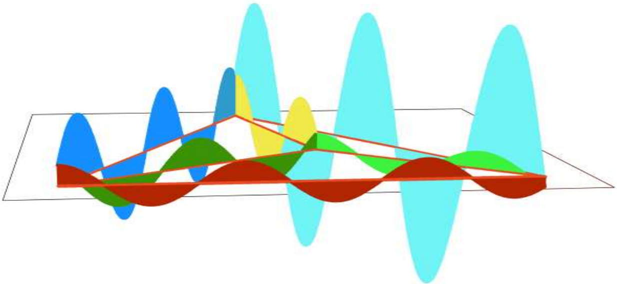
Nonlinear optics
Fibre optics

Metric Graph



- quasi one-dimensional trap or waveguide for ultra-cold atoms
- network of optical fibres

Waves on metric graphs



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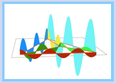

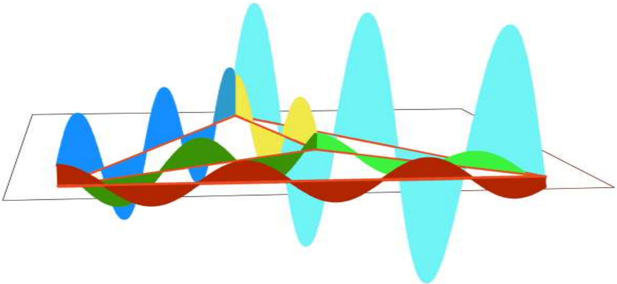

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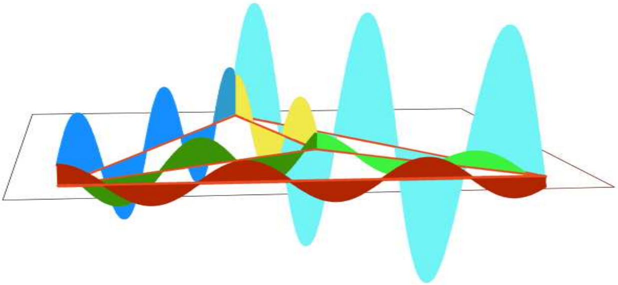

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- nonlinear waves: nonlinear quantum graph (?)
  Nonlinear Schrödinger/Gross-Pitaevski, Burger's, Korteweg-de Vries, Sine-Gordon

The stationary NLSE in one dimension

$$-\psi''(x) + g|\psi(x)|^2\psi(x) = E\psi(x)$$

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- integrable, stationary solutions are known (Carr *et al* , PRA 2001)

Classical dynamics picture for NLSE for $E > 0$

$$\tau = \sqrt{E}x, \qquad \psi(x) = \sqrt{\frac{2E}{|g|}}\, r(\tau) e^{i\theta(\tau)}, \qquad \sigma = \frac{g}{|g|}$$

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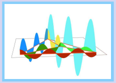

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  $\psi^* \psi' = \frac{2E^{3/2}}{|g|}\left(rp + i\ell\right)$
- If $\ell = 0$ one may choose ψ to be real

Integration of the stationary NLSE

Two constants of motion

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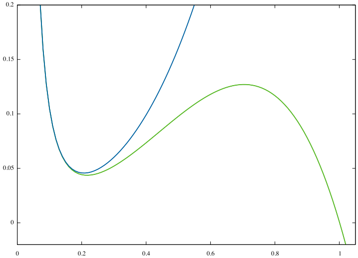

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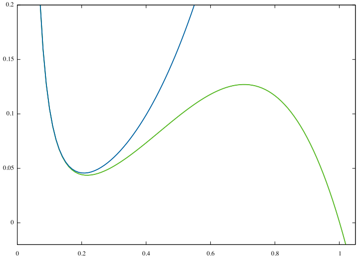

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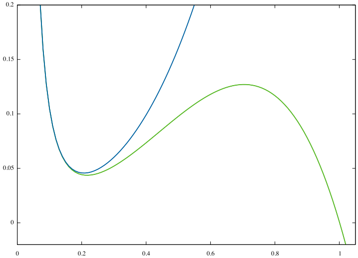

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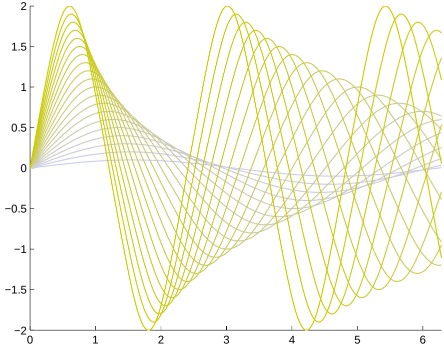
$$\theta - \theta_0 = \ell \int \frac{1}{r^2 \sqrt{2(H - V_{\mathrm{eff}}(r;\ell))}} dr$$

Amplitude dependence – attractive case

$\ell = 0$: $\rightarrow$ real solution

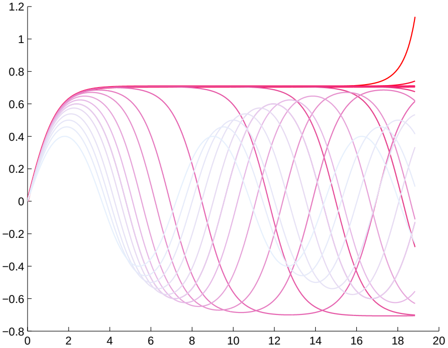
One parameter: $I_0 = \frac{|g|}{2E}\max\left(|\psi(x)|^2\right) = \max\left(r^2\right)$

$$\psi(x) = \sqrt{\frac{2E}{|g|}}\sqrt{\frac{I_0(I_0+1)}{2I_0+1}}\,\frac{\mathrm{sn}\left(\sqrt{(2I_0+1)E}x, \frac{I_0}{2I_0+1}\right)}{\mathrm{dn}\left(\sqrt{(2I_0+1)E}x, \frac{I_0}{2I_0+1}\right)}$$



Amplitude dependence – repulsive case

$$\psi(x) = \sqrt{\frac{2EI_0}{|g|}} \times \begin{cases} \operatorname{sn}\left(2\sqrt{(1-I_0)E}x, \frac{I_0}{1-I_0}\right) & \text{for } I_0 < 1/2 \\ \dfrac{\operatorname{sn}\left(2\sqrt{I_0 E}x, \frac{2I_0+1}{4I_0}\right)}{1+\operatorname{cn}\left(2\sqrt{I_0 E}x, \frac{2I_0+1}{4I_0}\right)} & \text{for } I_0 > 1/2 \end{cases}$$



NLSE on a metric graph

- vertices $i = 1, 2, \ldots, V$
- edges may either be finite bonds $b \equiv (i, j)$ that connect two vertices i and j
  or infinite leads $l = (i, \infty)$ that start a vertex i and go to infinity.

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- wave function $\Psi(x) = \{\psi_b(x_b), \psi_l(x_l)\}$ solve NLSE on the bonds and leads
- one may choose the coupling constant g differently on each edge.
  If $g = 0$ on an edge we call the edge (bond, lead) linear.

Matching conditions

- Consider one vertex with v incident edges
  and coordinates $x_1, x_2, \ldots, x_v$
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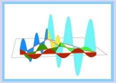

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- **Generalised Robin-type condition**

$$\sum_{i=1}^{v} \psi_i'(0) = \lambda \psi_0$$

where λ is real – vertex potential strength

Flux conservation

Robin-type matching conditions in terms of the classical dynamics picture (with $\psi_0 = \sqrt{2E/|g|}\, r_0 e^{i\theta_0}$)

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$$\sum_{j=1}^{v} p_j = \frac{\lambda}{\sqrt{E}}$$

$$\sum_{j=1}^{v} \ell_j = 0$$

Sum of fluxes (outgoing) at each vertex vanishes.

How to find the solution?

Consider a finite graph with V vertices and B bonds (no leads) and fix an energy $E > 0$.

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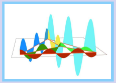

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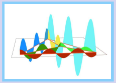

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1. Given a set of V complex numbers $\phi_i =$ find ℓ_b and $\mathcal{E}\mathrm{Ham}, b$ such that $\psi_b(x_b = 0) = \phi_i$ and $\psi_b(L_b) = \phi_j$.
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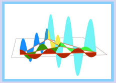

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2. The Robin-type matching conditions are then a set of v equations for v unknowns ϕ_i.
   Each solution $\{\phi_i\}$ yields a wave function Ψ on the nonlinear graph.

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$$\sum_b \int_0^{L_b} |\psi_b|^2 dx_b = N$$

- Fixing N implies energy quantisation.

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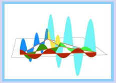

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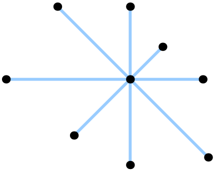
- For $N \to 0$ the equations reduce to simple matrix equations and yield an explicit secular equation.
  This may be exploited numerically and analytically.

Star graphs



- Only one vertex and no fluxes
  $\longrightarrow \ell_b = 0$ and $\psi_b = \psi_b^*$
- For fixed normalisation the Robin-type matching condition yields a secular equation for the 'spectrum'

Scattering from a nonlinear graph

- The physical relevance of the 'spectrum' is questionable apart from the ground state energy
- in a fibre optics application one may consider to inject a laser beam into a fibre network and measure response or transport

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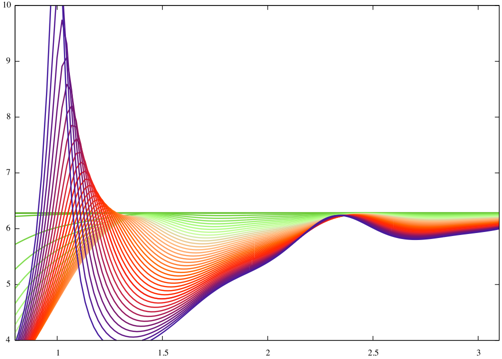
- at low intensities the first order is given by the scattering matrix of the corresponding linear quantum graph

Scattering phase and delay

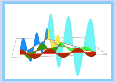
One lead attached to a star graph with incoming wave at energy $E = k^2$

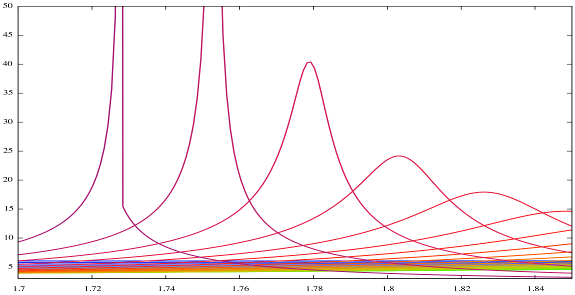
$$\psi_{\mathrm{lead}}(x) = 2\sqrt{\frac{I}{k}}\cos(kx + \delta/2) \qquad (1)$$

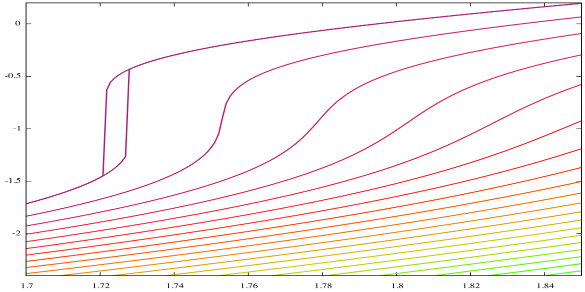
Scattering phase: $\delta = \delta(I, k)$
delay time: $\frac{\partial \delta}{\partial k}$



Multistabilities & hysteresis







Outlook

- transport in one-dimensional (periodic) graphs such as the chain, or a comb
- Ground state energy for BEC in a non-trivial trap, e.g. in figure eight shape
- (chaotic) scattering from nonlinear quantum graphs
- theory for time-dependent NLSE on graphs
- solitons in a non-linear graph