# Applications of Free Random Variables to Financial Analysis

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## Abstract

We apply the concept of *free random variables* (FRV), which is a noncommutative extension of probability calculus, to *doubly–correlated Gaussian Wishart* random matrix models, appearing for example in multivariate analysis of financial time series displaying both inter–asset cross–covariances $\mathbf{C}$ (such as in factor models) and temporal auto–covariances $\mathbf{A}$ (such as in models with heteroscedasticity, in VARMA processes, *etc.*).

## Correlated Gaussians

**We consider** a universe of $N$ financial assets ($i = 1, \ldots, N \to \infty$), sampled over $T$ time moments ($a = 1, \ldots, T \to \infty$): $R_{ia}$ is *e.g.* the demeaned logarithmic return. The simplest approximations: $\mathbf{R}$ is a *Gaussian* random matrix with the structure of covariances,

$$\langle R_{ia} R_{jb} \rangle = C_{ij} A_{ab}. \qquad (1)$$

The change of variables $\mathbf{R} = \sqrt{\mathbf{C}}\widetilde{\mathbf{R}}\sqrt{\mathbf{A}}$ gives uncorrelated Gaussians $\widetilde{\mathbf{R}}$.

**Problem:** *estimation* of $\mathbf{C}$ from historical time series. Marred by the measurement *noise*, quantified by $r \equiv N/T$. The Pearson estimator of $\mathbf{C}$,

$$\mathbf{c} = \frac{1}{T}\sqrt{\mathbf{C}}\widetilde{\mathbf{R}}\mathbf{A}\widetilde{\mathbf{R}}^{\mathrm{T}}\sqrt{\mathbf{C}}. \qquad (2)$$

## Cross– and Auto–Covariances

**Models of cross–covariances:** Typical spectra of $\mathbf{c}$: a few large eigenvalues ("factors") plus a "sea" of the Marčenko–Pastur distribution. The largest factor represents the market, the other factors — the industrial sectors.

**Models of auto–covariances:**

- *Heteroscedasticity* (volatility clustering) is a phenomenon of a stochastic time dependence of volatility $\sigma_{ia}$. These lagged correlations of volatility are not strong (a few percent), but persist over long periods of time. Typically modeled by a power–law decay, $\langle \sigma_a^2 \sigma_b^2 \rangle - \langle \sigma_a^2 \rangle \langle \sigma_b^2 \rangle \propto 1/|b-a|^{\nu}$ ($\nu \sim 0.2 \div 0.4$). An I–GARCH(1) (EWMA) model suggests a less realistic exponential decay $\mathrm{e}^{-|b-a|/\tau}$, yet widely exploited in the financial industry (RiskMetrics 1994, $\tau = 16.2$ days).
- Exponentially decaying *lagged correlations between the residuals* $R_{ia}/\sigma_{ia}$, albeit weak and short–ranged, are included in the new RiskMetrics 2006.

Other models in other contexts (*e.g.* in wireless communication).

## Free Random Variables

Voiculescu and Speicher's [4] *free random variables calculus* is a generalization of probability theory to *noncommutative random variables*, such as infinite (Hermitian) random matrices $\mathbf{X}$. It relies on the concept of *freeness*, which is noncommutative independence.

| **Classical probability:** | **Noncommutative probability (FRV):** |
|---|---|
| p.d.f., $P_X(x)$ | spectral density, $\rho_{\mathbf{X}}(\lambda)$ |
| characteristic function, $g_X(z) \equiv \langle \mathrm{e}^{\mathrm{i}zX} \rangle$ | Green's function, $G_{\mathbf{X}}(z) \equiv (1/N)\langle \mathrm{Tr} 1/(z\mathbf{1}_N - \mathbf{X}) \rangle$, or $M$–transform, $M_{\mathbf{X}}(z) = zG_{\mathbf{X}}(z) - 1$ |
| independence | freeness |
| **Addition of independent commutative random variables:** The logarithm of the characteristic function, $r_X(z) \equiv \log g_X(z)$, is additive, $$r_{X_1+X_2}(z) = r_{X_1}(z) + r_{X_2}(z). \qquad (3)$$ | **Addition of free noncommutative random variables:** The Blue's function, $G_{\mathbf{X}}(B_{\mathbf{X}}(z)) = B_{\mathbf{X}}(G_{\mathbf{X}}(z)) = z$, is additive, $$B_{\mathbf{X}_1+\mathbf{X}_2}(z) = B_{\mathbf{X}_1}(z) + B_{\mathbf{X}_2}(z) - \frac{1}{z}. \qquad (4)$$ |
| **Multiplication of independent r.v.:** Reduced to the addition problem via the exponential map, owing to $\mathrm{e}^{X_1}\mathrm{e}^{X_2} = \mathrm{e}^{X_1+X_2}$. | **Multiplication of free r.v.:** The $N$–transform, $M_{\mathbf{X}}(N_{\mathbf{X}}(z)) = N_{\mathbf{X}}(M_{\mathbf{X}}(z)) = z$, is multiplicative, $$N_{\mathbf{X}_1\mathbf{X}_2}(z) = \frac{z}{1+z} N_{\mathbf{X}_1}(z) N_{\mathbf{X}_2}(z). \qquad (5)$$ |

## The Main Result

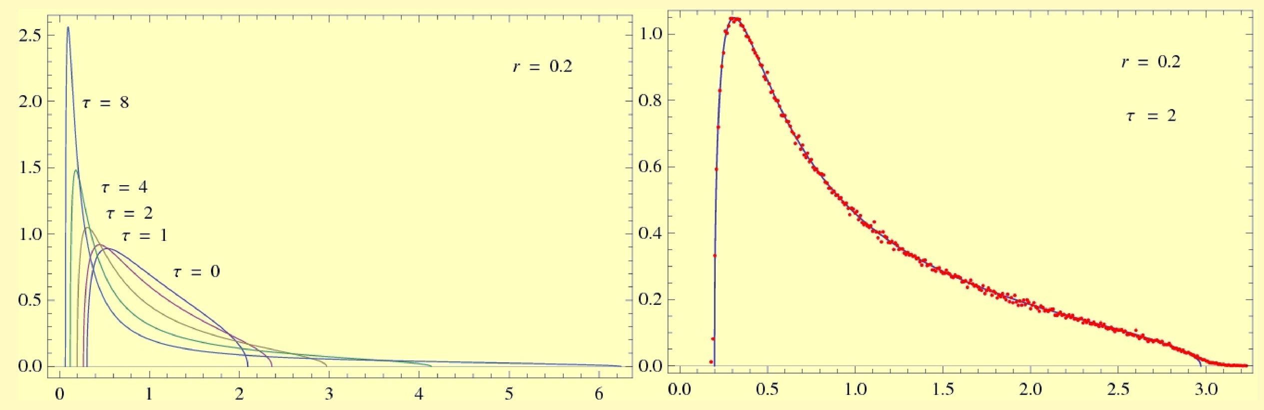
The FRV multiplication formula (5) leads via a back–of–an–envelope calculation to an equation for the $M$–transform of the Pearson estimator $\mathbf{c}$ (2) [1],

$$z = rMN_{\mathbf{A}}(rM)N_{\mathbf{C}}(M). \qquad (6)$$

## Examples

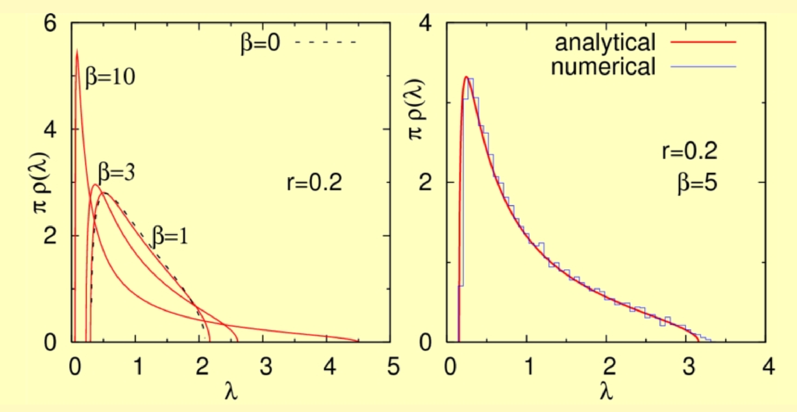
**Exponentially decaying auto–covariances:** $A_{ab} = \mathrm{e}^{-|a-b|/\tau}$; denote $\chi \equiv \coth(1/\tau)$. The main FRV formula (6) yields a 4–th order polynomial equation for $M \equiv M_{\mathbf{c}}(z)$ [1],

$$r^2M^4 + 2r(r - \chi z)M^3 + \left(z^2 - 2r\chi z + r^2 - 1\right)M^2 - 2M - 1 = 0. \qquad (7)$$



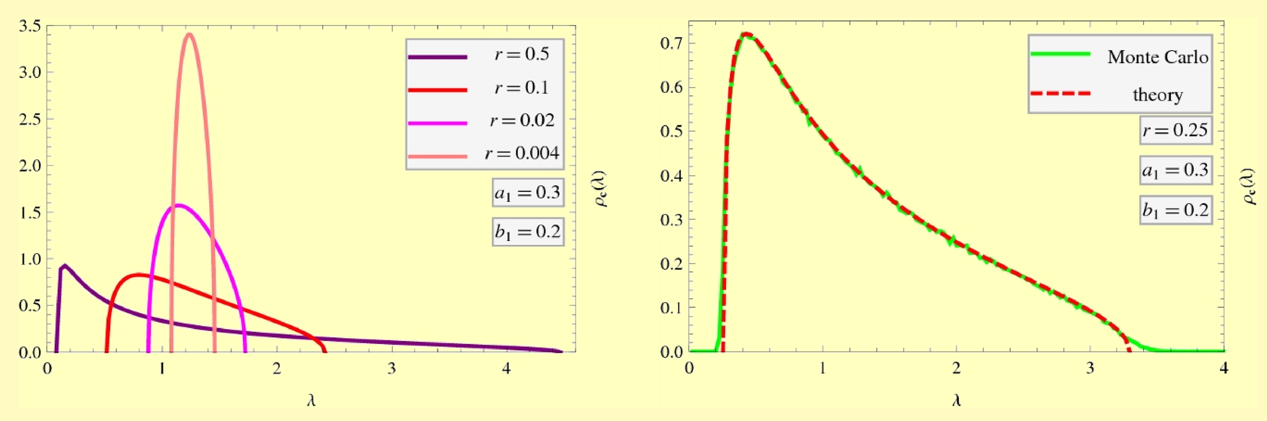
**Exponentially weighted moving average (EWMA):** $A_{ab} \equiv T\frac{1-\alpha}{1-\alpha^T}\alpha^{a-1}\delta_{ab}$, in the limit $\alpha \to 1^-$, such that $\beta \equiv T(1-\alpha)$ finite (in RiskMetrics 1994, $\alpha = 0.94$). The main FRV formula (6) yields an entangled equation for $M \equiv M_{\mathbf{c}}(z)$ [1],

$$z = \frac{r\beta(1+M)\left(\mathrm{e}^{\beta(1+rM)} - 1\right)}{\left(\mathrm{e}^{\beta} - 1\right)\left(\mathrm{e}^{\beta rM} - 1\right)}. \qquad (8)$$




**VARMA(1,1):** $R_{ia} - b_1 R_{i,a-1} = a_0\epsilon_{ia} + a_1\epsilon_{i,a-1}$ (a stochastic process), with $\epsilon_{ia}$ white noise. Its auto–covariance matrix $\mathbf{A}$ is an exponential decay with an additional constant term on the diagonal. The main FRV formula (6) yields a 6–th order polynomial equation for $M \equiv M_{\mathbf{c}}(z)$ [2],

$$r^4 a_0^2 a_1^2 \left(a_0^2 - a_1^2\right)^2 M^6 + \ldots = 0. \qquad (9)$$




Everywhere $\mathbf{C} = \mathbf{1}_N$, *i.e.*, trivial cross–covariances. The graphs show the mean spectral density $\rho_{\mathbf{c}}(\lambda)$ inferred from FRV equations (7), (8), (9) (left), and their Monte Carlo numerical confirmation (right).

## Prospects

Incorporation of non–Gaussian effects:

- Free Lévy (stable w.r.t. FRV) vs. standard Lévy [3].
- Stochastic volatility models (modeling of heteroscedasticity): $\mathbf{C}$ and $\mathbf{A}$ random instead of constant ("random deformations"); our main formula (6) remains usable.

Inclusion of a nontrivial cross–correlation structure $\mathbf{C}$ inferred from factor models. Problem: Large ($\sim N$) eigenvalues (the market and industrial sectors) of $\mathbf{C}$ not compatible with FRV.

More general correlation structure than just $\mathbf{C}$ and $\mathbf{A}$ (1). Goal: Taking into account the Epps effect.

## References

[1] Burda Z., Jarosz A., Jurkiewicz J., Nowak M. A., Papp G., Zahed I., *Applying Free Random Variables to Random Matrix Analysis of Financial Data*, submitted to Quantitative Finance.

[2] Burda Z., Jarosz A., Nowak M. A., Snarska M., *A Free Random Variables Approach to the VARMA Processes*, submitted to New J. Phys.

[3] Burda Z., Jarosz A., Jurkiewicz J., Nowak M. A., Papp G., Zahed I., *Free Lévy–Wishart Random Matrices with Spatial and Temporal Correlations*, in preparation.

[4] Voiculescu D. V., Dykema K. J., Nica A., *Free Random Variables*, Am. Math. Soc., Providence, 1992. Speicher R., Math. Ann. **298** (1994) 611.

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