Gaussian Random Waves in Elastic Medium

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Abstract

Similar to Berry conjecture for quantum chaos we consider elastic analogue which incorporates longitudinal and transverse random waves. Based on that we derive the intensity correlation function of elastic displacement field. Comparison to numerics in a quarter Bunimovich stadium demonstrates a good agreement. We also consider nodal points (NPs) $u = 0,\ v = 0$ of the in-plane random vectorial displacement field $\mathbf{u} = (u, v)$. We derive the mean density and correlation function of NPs. Consequently, we derive the distribution of the nearest distances between NPs.

1. Introduction

Attracting interest in the field of wave chaos, elastomechanical systems are being studied analytically, numerically, and experimentally [1]. Weaver first measured the few hundred lower eigen frequencies of an aluminum block and worked out the spectral statistics [2]. Statistical properties of eigen functions describing standing waves in elastic billiards were first reported by Schaadt *et al.* [3]. The authors measured the displacement field of several eigen modes of a thin plate shaped as a Sinai stadium. Due to a good preservation of up-down symmetry in the case of thin plates there are two types of modes. The flexural modes with displacement perpendicular to the plane of the plate are well described by the scalar biharmonic Kirchhoff-Love equation. In this case a perfect agreement with theoretical prediction for both intensity statistical distribution and intensity correlation function was found. However in the case of in-plane displacements described by the vectorial Navier-Cauchy equation an agreement between the intensity correlator experimental data and the theory was not achieved [3].

The aim of this work is to present an analogue of Berry conjecture for elastic vibrating solids and derive the amplitude and intensity correlators with corresponding comparison to numerics. We propose here a simple and physically transparent approach based on random superposition of traveling plane waves (Gaussian random wave). We restrict ourselves to the two-dimensional case. However, the method can be easily generalized for the three-dimensional case.

The simplest way to construct Gaussian random field is to use a random superposition of N plane waves. Thus we come to Berry conjecture in the form

$$\phi(\mathbf{x}) = \sqrt{\frac{1}{N}}\sum_{n=1}^{N}\exp[(i(\theta_n + \mathbf{k_n x})], \qquad (1)$$

where the phases θ_n are distributed uniformly in range $[0, 2\pi)$ and all the amplitudes are equal (one could assume random independent amplitudes, without any change in the results). The wave vectors $\mathbf{k}_n$ are uniformly distributed on a d-dimensional sphere of radius k. It follows now from the central limit theorem that both $\Re\{\phi\}$ and $\Im\{\phi\}$ are independent Gaussian variables. In a finite-size billiard the conjecture is viewed as a sum of many standing waves, that is simply real or imaginary part of function (1).

In our case one has to construct a Gaussian random wave (GRW) describing acoustic in-plane modes. The mode shapes are given by the two-dimensional Navier-Cauchy equation

$$\mu\nabla^2\mathbf{u} + (\lambda+\mu)\nabla(\nabla\mathbf{u}) + \rho\Omega^2\mathbf{u} = 0 \qquad (2)$$

where $\mathbf{u}(x, y)$ is the displacement field in the plate, λ, μ are the material dependent Lamé coefficients, and ρ is the density. Introducing elastic potentials ψ and $\mathbf{A}$ with the help of the Helmholtz decomposition the displacement field $\mathbf{u}$ could be written,

$$\mathbf{u} = \mathbf{u}_l + \mathbf{u}_t, \quad \mathbf{u}_l = \nabla\psi, \ \mathbf{u}_t = \nabla\times\mathbf{A} \qquad (3)$$

Eq. (2) reduces to two Helmholtz equations for the elastic potentials

$$-\nabla^2\psi = k_l^2\psi, \quad -\nabla^2\mathbf{A} = k_t^2\mathbf{A}. \qquad (4)$$

Here $k_l = \omega/c_l$, $k_t = \omega/c_t$ are the wave numbers for the longitudinal and transverse waves, respectively and $\omega^2 = \rho\Omega^2/E$, where E is Young's modulus. In the two-dimensional case potential $\mathbf{A}$ has only one none-zero component A_z and the dimensionless longitudinal and transverse sound velocities $c_{l,t}$ are given by

$$c_l^2 = \frac{1}{1-\sigma^2}, \quad c_t^2 = \frac{1}{2(1+\sigma)}, \qquad (5)$$

where σ is Poisson's ratio [4]. E and σ are functions of the Lamé coefficients [4].

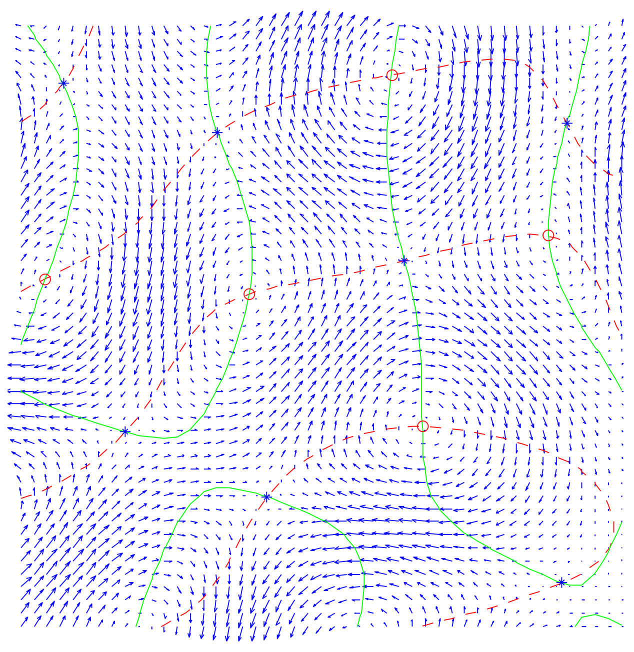

Figure 1: Transverse polarized random wave field.

Our conjecture is that both elastic potential be statistically independent Gaussian random waves (1). We write the potentials in the following form

$$\psi(\mathbf{x}) = \frac{1}{ik_l}\sqrt{\frac{\gamma}{N}}\sum_{n=1}^{N}\exp[i(\mathbf{k}_{ln}\mathbf{x} + \theta_{ln})], \qquad (6)$$

$$A_z(\mathbf{x}) = \frac{1}{ik_t}\sqrt{\frac{1-\gamma}{N}}\sum_{n=1}^{N}\exp[i(\mathbf{k}_{tn}\mathbf{x} + \theta_{tn})], \qquad (7)$$

where θ_{ln}, θ_{tn} are statistically independent random phases. The wave vectors $\mathbf{k}_{ln}$ and $\mathbf{k}_{tn}$ are uniformly distributed on circles of radii k_l and k_t respectively. Prefactors $\sqrt{\gamma}$ and $\sqrt{1-\gamma}$ are chosen from the normalization condition $\langle\mathbf{u}^{\dagger}\mathbf{u}\rangle = 1$, and $\langle\ldots\rangle$ means average over the random phase ensembles.

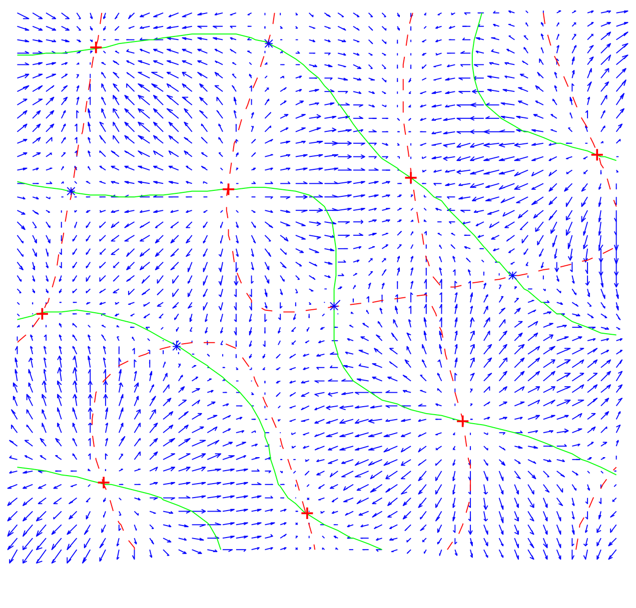

Figure 2: Longitudinal polarized random wave field.

2. Correlation function

Now we can calculate the intensity correlation functions $P(s) = \langle I\mathbf{x}+\mathbf{s})I(\mathbf{x})\rangle$ where the intensity $I = |\mathbf{u}|^2$ is proportional to the elastic energy of the in-plane oscillations. For the in-plane chaotic GRW of the form $a_l\psi_l + a_t\psi_t$ Schaadt *et al.* [3] derived the intensity correlation function as

$$P(s) = 1 + 2[a_l^2 J_0(k_l s) + a_t^2 J_0(k_t s)]^2. \qquad (8)$$

Our calculations give a different result

$$P(s) = 1 + [(\gamma J_0(k_l s) + (1-\gamma)J_0(k_t s)]^2 + [(\gamma J_2(k_l s) - (1-\gamma)J_2(k_t s)]^2 \qquad (9)$$

Although the first term in (9) corresponds to (8) there is a different term consisted of the Bessel functions $J_2(x)$. The mathematical origin of deviation is that formula (9) contains the contributions of the components of the wave vectors $\mathbf{k}_l$ and $\mathbf{k}_t$ via space derivatives.

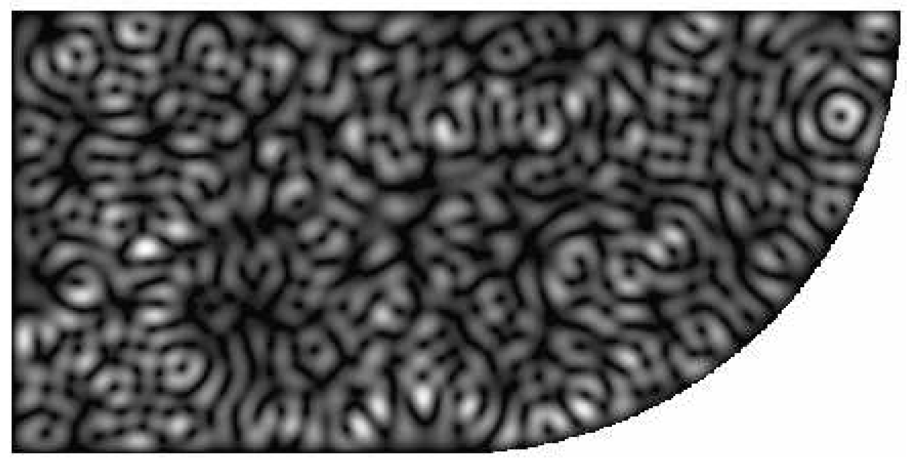

Figure 3: The energy density of a chaotic eigen state.

Waves propagate freely inside the billiard, that is, the longitudinal and transverse components are decoupled. The estimate of energy partition between weakly coupled longitudinal and transverse modes was first given by Weaver in [4]. We use a different way to approach this problem. In a billiard wave conversion occurs at the boundary according to Snell's law

$$c_l\sin(\theta_t) = c_t\sin(\theta_l), \qquad (10)$$

The reflection amplitudes for each event of the reflection can be easily found from wave equation (2). At first we consider more easy case of the Dirichlet BC (the boundary is clamped). Approximating the boundary as the straight lines for the wavelengths much less than the radius of curvature we have for the reflection amplitudes

$$t_{ll} = \frac{\cos(\theta_t)\cos(\theta_l) - \sin(\theta_t)\sin(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}, \quad t_{lt} = \frac{2\sin(\theta_l)\cos(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)},$$
$$t_{tl} = \frac{2\sin(\theta_t)\cos(\theta_t)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}, \quad t_{tt} = \frac{\cos(\theta_t)\cos(\theta_l) - \sin(\theta_t)\sin(\theta_l)}{\cos(\theta_t)\cos(\theta_l) + \sin(\theta_t)\sin(\theta_l)}. \qquad (11)$$

Next, we assume that all directions of waves are statistically equivalent. Then we have for the energy density of reflected wave

$$\rho_{out} = \gamma(\overline{T}_{ll} + \overline{T}_{lt}) + (1-\gamma)(\overline{T}_{tt} + \overline{T}_{tl}), \qquad (12)$$

where

$$\overline{T}_{ij} = \frac{1}{\pi}\int_0^{\pi} t_{ij}^2 d\theta_i, \ i = l, t.$$

Substituting Eq. (11) into Eq. (12) one can obtain after elementary calculations

$$\overline{T}_{ll} = 1 - \tfrac{c_t}{c_l}I_1, \ \overline{T}_{lt} = I_2,$$
$$\overline{T}_{tt} = 1 - \tfrac{2}{\pi}\arcsin\tfrac{c_t}{c_l} + \left(\tfrac{c_t}{c_l}\right)^3 I_1, \ \overline{T}_{tl} = \tfrac{2}{\pi}\arcsin\tfrac{c_t}{c_l} - \left(\tfrac{c_t}{c_l}\right)^2 I_2. \qquad (13)$$

We do not present here integrals I_1, I_2, since after substitution of (13) into (12) they cancel each other. The equality $\rho_{in} = 1 = \rho_{out}$ gives a very simple evaluation

$$\gamma = \frac{c_t^2}{c_t^2 + c_l^2}. \qquad (14)$$

which is the same as given in [4]. The next remarkable result is that although the reflection amplitudes for the free BC have the form different from (11), the evaluation of γ has the same form as for the fixed BC i.e. the result does not depend on either the free BC or the clamped BC is applied.

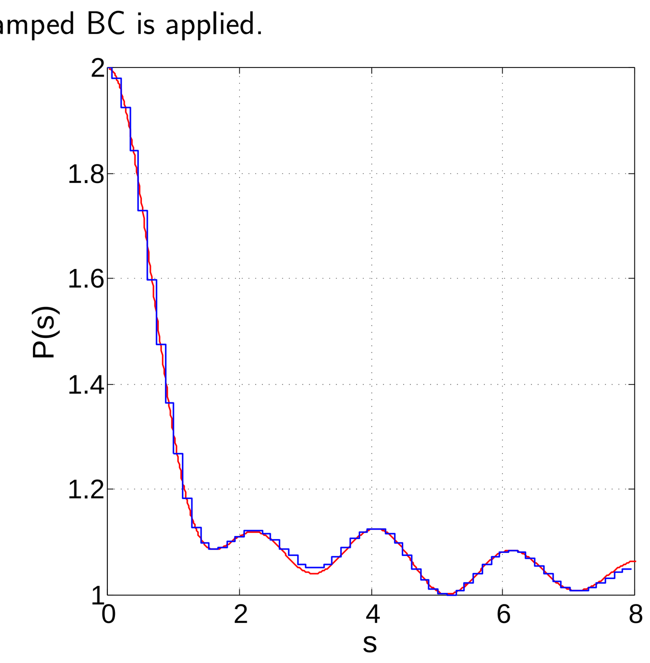

Figure 4: Numerically computed correlation function in comparison with Eq. (9) shown by red line.

3. Nodal points statistics

Next, we derive correlation function of nodal points (NPs) of the random elastic field and statistics of the nearest distances between them similar to that considered in [5,6] for quantum chaos. For closed elastic plate (billiard) the vectorial in-plane displacements $\mathbf{u}(x, y) = (u(x, y)v(x, y))$ have NPs $u(x_0, y_0) = 0, \quad v(x_0, y_0) = 0$ at the point $\mathbf{x}_0 = (x_0, y_0)$ (see Fig. 1 and Fig. 2). NPs of vectorial field are specified by the Poincaré index (topological charge)

$$q = sign(\det M_{\mathbf{x}_0}) = sign(\lambda_1\lambda_2), \ M = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}, \qquad (15)$$

where $\lambda_{1,2}$ are eigenvalues of matrix M at NP $\mathbf{x}_0$. Depending on these eigenvalues NPs split on the four types: 1) centers for imaginary $\lambda_{1,2}$ with the index $q = 1$; 2) knots for real $\lambda_{1,2}$ with the same sign and $q = 1$; 3) focuses for complex $\lambda_1 = \lambda_2^*$ with $q = 1$; and 4) saddles for real $\lambda_{1,2}$ with opposite sign and $q = -1$. Eigenvalues of matrix M are

$$\lambda_{1,2} = \frac{u_x + v_y}{2} \pm \sqrt{(\frac{u_x + v_y}{2})^2 - \det M} = \frac{u_x + v_y}{2} \pm \sqrt{D}/4 \qquad (16)$$

where we introduced $D = (u_x + v_y)^2 - 4\det M$. Therefore, for $D > 0$ NP can be classified as a knot, while for $D < 0$ we have a focus (center for particular case $u_x + v_y = 0$). At last for $M < 0$ NP is a saddle.

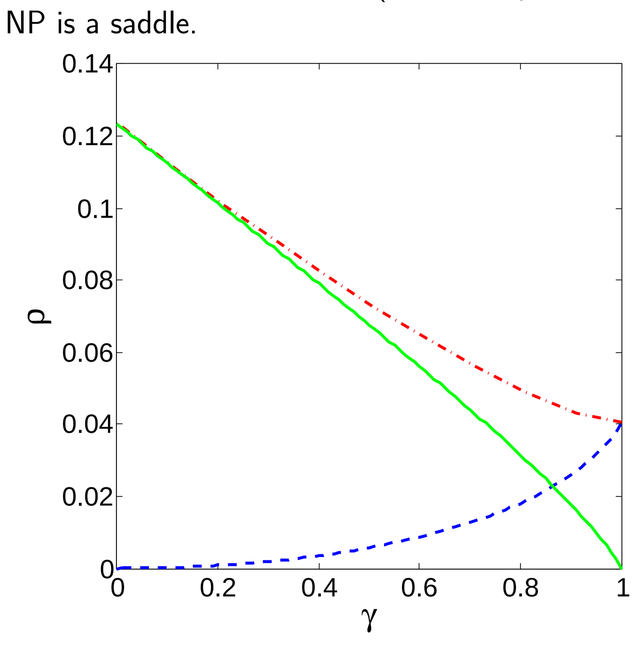

Figure 5: The density of saddles, red dash-dot line; focuses; green solid line; and knotes, blue dotted line. The material parameter $\sigma = 0.345$.

Averaging over Gaussian random fields makes possible to derive the density of NPs in elastic vectorial field. For the NPs density we have

$$\rho = \langle\delta(u)\delta(v)|M|\rangle, \qquad (17)$$

where the Jacobian $|M|$ is found from Eq.(15). We also derived the densities of all nodal points types Fig. 5.

The corresponding correlation function is [5,6]

$$G(s) = \frac{1}{\rho^2}\langle\delta(u)\delta(v)|M|\delta(u_s)\delta(v_s)|M_s|\rangle, \qquad (18)$$

where for brevity we omitted coordinate arguments of values except index s which implies the distance between the points $\mathbf{x}$ and $\mathbf{x}+\mathbf{s}$. Similarly, the charge correlation function [5,6] gives the correlation of density, but weighted with their charge q,

$$G_q(s) = \frac{1}{\rho^2}\langle\delta(u)\delta(v)M\delta(u_s)\delta(v_s)M_s\rangle. \qquad (19)$$

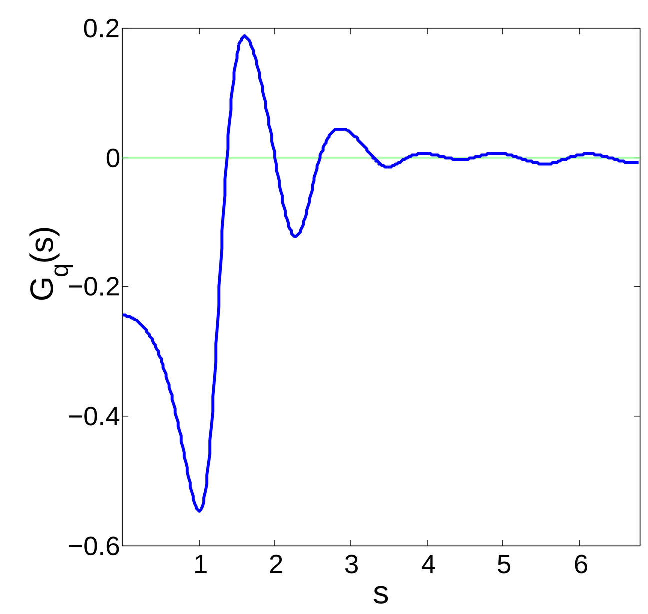

Figure 6: Topological charge correlation function

The distribution of the nearest distances between NPs can be derived from the density correlation function (18). Following [6] we use Poisson approximation for which the distribution is

$$f(s) \approx 2\pi s G(s)\exp(-\langle n(s)\rangle), \qquad (20)$$

where the mean number of NPs inside the circle of radius s around given one is

$$\langle n(s)\rangle = 2\pi\rho\int_0^s rG(r)dr, \qquad (21)$$

with ρ as mean density of NPs.

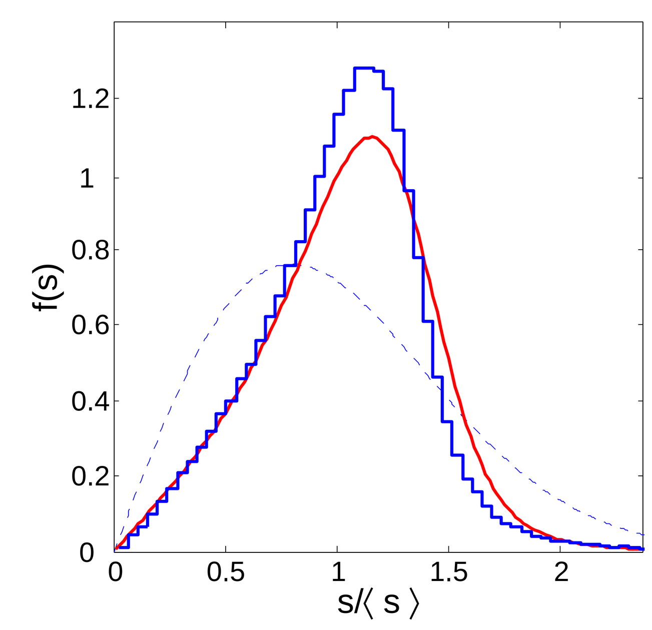

Figure 7: Histogram of the distribution function of the nearest distances between NPs for the random in-plane elastic wave field; $\sigma = 0.345$. Solid line shows the distribution function calculated in the Poisson approximation (20). Dashed line shows the case of uniformly distributed and completely random points [7] $f(x) = \frac{\pi x}{2}\exp(-\pi x^2/4)$.

4. Summary

Based on random waves theory we investigated in-plane eigen modes of chaotic elastic billiards. With the help of the Helmholtz decomposition we succeeded to present an adequate description of intensity correlation properties of Gaussian random elastic waves. Moreover, in case of billiards we found that double ray splitting at the billiard boundary affects the partition of energy between longitudinal and transverse waves. The results obtained were verified numerically. Next, we considered the statistical properties of nodal points in random elastic displacement field. We investigated their topological properties, derived the charge-charge correlation function, and numerically computed the density-density correlation function. These results were the starting point for finding the distribution of the nearest distances between the nodal points. We shown that our approximate result demonstrates a good accuracy. For further reading see [8,9].

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